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Granular flow in a rotating cylindrical drum

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Abstract. – Flow of granular material in a partially filled rotating cylinder is studied using a boundary layer approach. Simple equations are derived to describe granular dynamics in a transverse plane. A small parameter is recognized, and the derived equations are solved analytically in the first-order approximation of the small parameter. Cascading layer thickness, mean velocity along the layer and profile of the free surface are determined in a closed analytical form.

Dynamics, mixing and separation of granular materials in a partially filled rotating cylindrical drum have been the subject of numerous experimental and theoretical investigations (see, *e.g.*, [1]-[3]). However only a few studies attempted to describe a continuous granular flow in a transverse plane of a rotating drum [1], [3]. In the present work a boundary value approximation is used to describe a two-dimensional granular flow. In the developed approach the transition from solid to liquid-like behavior is considered as a key process for specifying the granular dynamics.

Let us consider a cylindrical drum of radius R which rotates with a constant angular velocity ω (fig. 1). The drum is partially filled with a granular material of constant bulk density ρ, and the length of the free surface is $2L$. The free surface is inclined at an angle of internal friction of the granular material μ. The flow in the drum can be described as follows. Particles rotate with the bulk of the granular material from right to left and fall down into a thin cascading layer at the free surface. We assume that the angular velocity ω is sufficiently large to produce continuous avalanches but the centrifugal forces are much smaller than the gravity force. The height of the free surface is $h(x)$, the thickness of the cascading layer is $\delta(x)$ and the horizontal and vertical components of velocity in the layer are $v_x(x, y)$ and $v_y(x, y)$, respectively. In the present work the boundary layer approximation is used in order to describe a velocity

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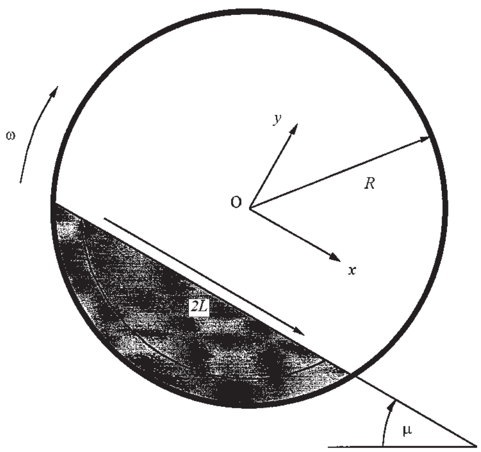

Fig. 1. – Schematic view of the flow in a rotating cylindrical drum and the coordinates system.

distribution across the layer. We assume that the horizontal velocity v_x has a constant profile in the transverse direction:

$$v_x(x,y) = u(x) f\left(\frac{y}{\delta}\right) ,$$

where $u(x)$ is the mean velocity and the function $f(\frac{y}{\delta})$ satisfies the conditions: $f(0)=0$,

$$\int_0^1 f(\eta)\mathrm{d}\eta = 1 .$$

In order to simplify the analysis let us rescale the independent variables as follows:

$$(x,y) = L(\bar{x}, \varepsilon\bar{y}), \quad (v_x, v_y) = \sqrt{\varepsilon g L}(\bar{v}_x, \varepsilon\bar{v}_y), \quad h = \varepsilon L \bar{h}, \quad \delta = \varepsilon L \bar{\delta} ,$$

where g is the acceleration of gravity and $\varepsilon \propto \delta/L$ is a small parameter whose physical meaning is elucidated below. The variables with overbars are dimensionless and are assumed to be of the order of 1. Such a scaling was chosen to balance different terms in the equations of mass and momentum conservation. Hereafter in order to simplify the notations the overbars will be omitted.

The equation of mass conservation for the layer reads

$$\rho\varepsilon^{3/2}\sqrt{gL}\frac{\mathrm{d}}{\mathrm{d}x}(u\delta) = -\rho\omega L x , \tag{1}$$

where the term on the right side accounts for the particles inflow into the cascading layer due to the rotation of the drum. Integrating the above equation with the boundary conditions $u\delta = 0$ at $x = \pm 1$ we obtain the expression for the material flux $q(x)$ at the free surface:

$$q(x) = u(x)\delta(x) = \frac{\Omega}{2}(1 - x^2) , \tag{2}$$

where $\Omega = \varepsilon^{-3/2}\omega\sqrt{L/g}$ is the dimensionless angular velocity which is assumed to be of the order of 1, *i.e.* we suppose that the rotational Froude number $Fr^2 = \omega^2 R/g \approx \omega^2 L/g$ is small. Note that eq. (2) was obtained previously by many authors [1], [3], [4].

Consider an interface between the flowing layer and the bulk of the granular material which can be defined as a thin region where the granular material experiences intensive plastic deformation. The shear and normal stresses in the layer and in the bulk are τ^+_{xy}, τ^+_{yy}, τ^-_{xy} and τ^-_{yy}, respectively. According to experimental and numerical observations [2], [5] the transition from the solid to liquid-like behavior is governed by a Mohr-Coulomb failure criterion, *i.e.* at the interface the ratio between a shear and a normal stresses is constant:

$$\tau^-_{xy}/\tau^-_{yy} = \tan(\mu)\,, \tag{3}$$

and the bulk is interpreted as a rigid-plastic body. Since the cascading layer is thin, we assume that the normal stress at the interface is equal to the weight of the granular layer

$$\tau^+_{yy} = \tau^-_{yy} = \varepsilon\rho g R\delta\,. \tag{4}$$

Since granular material in the flowing layer is well fluidized, the well-known Bagnold law [6] for the shear stress can be used:

$$\tau^+_{xy} = \chi(\lambda)\rho_{\text{particles}} D^2_{\text{particles}} \left|\frac{\partial v_x}{\partial y}\right|^2 ,$$

where $\chi(\lambda)$ is a function of the interparticle spacing and the derivative $\partial v_x/\partial y$ is calculated at the interface. Thus, for a given velocity profile $f(\frac{y}{\delta})$ the latter equation yields

$$\tau^+_{xy} = \rho D^2 \frac{gA}{\varepsilon L}\left(\frac{u}{\delta}\right)^2 , \tag{5}$$

where A is a constant which depends on the particles properties and the velocity distribution across the layer. Assuming that the shear stress is continuous at the interface, *i.e.* $\tau^+_{xy} = \tau^-_{xy}$ and combining eqs. (3), (4) and (5), we obtain

$$\rho D^2 \frac{gA}{\varepsilon L}\left(\frac{u}{\delta}\right)^2 = \varepsilon\rho g L\delta\,. \tag{6}$$

Now we can set $\varepsilon^2 = AD^2/L^2$, *i.e.* a small parameter $\varepsilon \propto D/L$ is proportional to the ratio of the particles diameter to the length of the free surface. Thus eq. (6) implies that

$$\delta^3 = u^2\,. \tag{7}$$

The latter result means that friction at the base of the layer determines its thickness by mobilizing new grains from the stationary bulk into the motion when the velocity of the layer increases and by extracting the grains from the layer into the stationary bulk when velocity decreases. Substituting eq. (7) into eq. (2), we obtain the expressions for the velocity distribution along the layer and the thickness of the layer:

$$\begin{aligned} u(x) &= \left[\frac{\Omega}{2}(1-x^2)\right]^{3/5} , \\ \delta(x) &= \left[\frac{\Omega}{2}(1-x^2)\right]^{2/5} . \end{aligned} \tag{8}$$

Notably, the equation of mass conservation and the condition of stress continuity at the bulk-flowing layer interface together with the boundary layer approximation are sufficient to specify a velocity field in the drum. The granular dynamics is governed by the conditions at the interface which relates the thickness of the layer with its velocity.

The only undetermined function is a profile of the free surface of the flowing granular material. In order to complete the solution of the problem, consider the equation of momentum conservation averaged across the flowing layer:

$$\alpha\varepsilon^2\rho g L \frac{\mathrm{d}}{\mathrm{d}x}(\delta u^2) = \varepsilon\rho g L\delta\left[\sin(\mu) - \varepsilon\cos(\mu)\frac{\mathrm{d}h}{\mathrm{d}x}\right] - \varepsilon\tan(\mu)\cos(\mu')\rho g L\delta\,, \tag{9}$$

where $u^2\int_0^1 f^2\mathrm{d}\eta = \alpha u^2$.

Thus there are two adjustable parameters in the model: the small parameter ε which accounts for the stress at the interface and a parameter α which describes a profile of the velocity and, therefore, depends on the friction inside the cascading layer. For simplicity we assume that the velocity profile in the transverse direction $f(\frac{y}{\delta})$ is blunt and $\alpha = 1$ but the analysis can be done for any other profile. The first term in the right side of eq. (9) is a gravitational force which weakly depends on the variation of h, *i.e.* the term $\mathrm{d}h/\mathrm{d}x$ describes the variation of the mean pressure along the layer. The second term on the right-hand side of eq. (9) is the frictional force which is exerted by the stationary bed at the cascading layer.

Let us consider the friction force in more details. Bagnold's law [6] prescribes that the shear stress is proportional to the square of the strain rate. On the other hand chute flows experiments indicate that the shear stress exhibits a weak rate dependence [7], [8]. This discrepancy can be explained as follows [8]. Although the shear stress in the cascading layer depends on the shear rate, it follows from eq. (3) that the ratio between the shear and the normal stresses at the interface is rate-independent. Since the normal stress is overburden pressure, *i.e.* $\tau_{yy}\propto\delta$, the shear stress is also proportional to the layer thickness: $\tau_{xy}\propto\delta$ and the friction force has the Coulomb-like rate-independent form. The angle $\mu' = \mu - \tan^{-1}(\mathrm{d}(h-\delta)/\mathrm{d}x)$ is the angle of inclination of the bed-layer interface. Expanding it in power series of the small parameter ε we obtain the correction to the friction force due to the slope change

$$\cos(\mu') \approx \cos(\mu) + \varepsilon\sin(\mu)\frac{\mathrm{d}(h-\delta)}{\mathrm{d}x}\,.$$

In eq. (9) the gravitational and the frictional terms are of the order of ε and they are mutually canceled. Therefore the scaling $u\propto\sqrt{\varepsilon}$ assumed above is necessary in order to balance the particles inertia with the variation of the pressure along the cascading layer. Thus eq. (9) reads

$$\frac{\mathrm{d}}{\mathrm{d}x}(\delta u^2) - \sin(\mu)\delta\frac{\mathrm{d}}{\mathrm{d}x}(\delta) = -\delta(\cos(\mu)+\sin(\mu))\frac{\mathrm{d}h}{\mathrm{d}x}\,.$$

Using eq. (8) one can integrate the latter equation to calculate the shape of the free surface:

$$h(x) = \frac{\frac{8}{7}\left(\frac{\Omega}{2}\right)^{\frac{6}{5}}[C_1 - (1-x^2)^{\frac{6}{5}}] - \sin(\mu)\left(\frac{\Omega}{2}\right)^{\frac{2}{5}}[C_2 - (1-x^2)^{\frac{2}{5}}]}{\cos(\mu)+\sin(\mu)}\,, \tag{10}$$

where the integration constants $C_1\approx 1.26$, $C_2\approx 1.46$, are determined from the mass conservation condition $\int_{-1}^{1} h\mathrm{d}x = 0$. Equation (10) shows that the height of the free surface can have bimodal or unimodal profile depending on the magnitude of the angular velocity Ω. When $\Omega \le 2(\frac{7\sin(\mu)}{24})^{5/4}$ the free-surface height has the single maximum at $x=0$, while for $\Omega \ge 2(\frac{7\sin(\mu)}{24})^{5/4}$ there are minimum at $x=0$ and two symmetrically located maxima. The latter behavior of a free-surface height is in a qualitative compliance with the available experimental results. Indeed, concave, flat and S-shaped free-surface profiles are observed in rotating cylindrical drums depending on the flow regimes [1]-[4]. Note that the height gradient $\mathrm{d}h/\mathrm{d}x$ is singular near the walls of the container.

In summary, we have derived the averaged equations for granular flow in the transverse plane of a rotating drum. The two adjustable parameters of the model depend on the magnitude of friction in the flowing layer. The solutions for the cascading layer thickness, averaged velocity and the free-surface profile are obtained in a closed analytical form.

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