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Kinematics of the mixing of granular material in slowly rotating containers

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Abstract. – The mixing of granular material in a two-dimensional slowly rotating container is studied using a kinematic approach. A simple model is proposed to estimate the mixing capacity of a mixer when the particles diffusivity is small. Different geometries of the container are examined numerically. The mixing rate index for the case of a circular drum is derived analytically. The optimum shape of the container and the optimum filling level are discussed.

Powder mixing in a rotating drum is a common industrial process which has been studied extensively during the last years [1-6]. Although a full understanding of solids mixing mechanisms in a rotating drum is still missing, it was recently demonstrated that even simple geometrical considerations can provide very helpful information [1,7-9] about the characteristic mixing time and the optimal filling level of the container. Different regimes of avalanche mixing (diffusion with convection and rapid mixing) which depend on a filling level in a rotating drum were investigated in [8]. In the present study we propose a kinematic model of powder mixing by rotationally induced avalanches.

Consider a two-dimensional convex mixer $r = r(\phi)$ (see fig. 1) partially filled with a granular material with constant bulk density. The container slowly rotates around a horizontal axis with a constant angular velocity $\dot{\theta} = 1$. Thus, the angle of rotation can be used instead of the time variable. The free surface of the granular material is supposed to be flat and it is inclined at a constant angle of repose. The height of the free surface is $z = h(\theta)$. Each particle rotates with the bulk of the material and when it reaches the free surface it falls down and sediments at the bulk at a new position.

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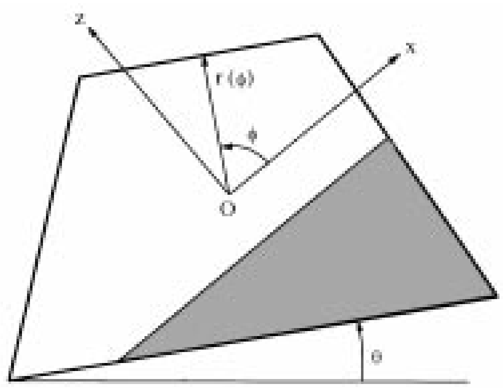

Fig. 1

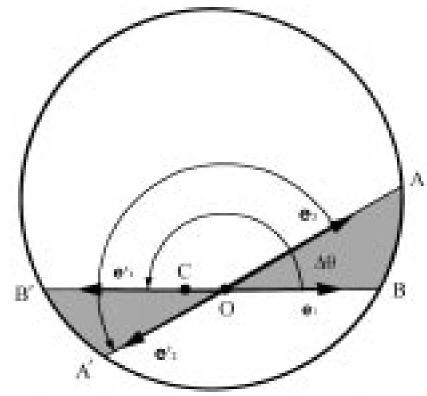

Fig. 2

Fig. 1. – Rotating container partially filled with a granular material. The X-axis is parallel to the free surface of the material, *i.e.* the angle of repose of the granular material plays no role in our considerations.

Fig. 2. – Scheme of avalanches: the granular bulk is rotating with the container by a small angle $\Delta\theta$ and after that the upper wedge AOB falls down to the new position $A'OB'$.

In order to simplify the analysis, let us replace the real avalanche dynamics by a sequence of regular avalanches. Assume that the free surface is rotated by a small angle $\Delta\theta$ and after that an avalanche occurs (fig. 2). Thus, after this rotation the granular material which occupies the upper triangle AOB fills the lower triangle $A'OB'$, *i.e.* AOB is mapped onto $A'OB'$. In order to complete the description of the flow, one must specify the location where the individual particle sediments with respect to the avalanching wedge. Metcalfe *et al.* [1, 7] developed a model with perfect mixing in the avalanches based on their experiments with small containers. Thus the dynamics of avalanche mixing can be described by a sequence of linear mappings. Recently, this model was extended to the case of the continuous flow in a circular drum [8, 9].

Note that real systems have a finite mixing rate at the free surface which varies in a wide range and depends on the flow regime and properties of the granular material [10]. When the size of the container becomes much larger than the typical diffusion scale, the perfect-mixing assumption cannot be used. In the present study we consider an opposite simple case with a negligibly small diffusion inside the avalanching wedge of granular material. Thus, grains are only advected by a macroscopic flow similar to the case of fluid mixing [11].

Consider the triangles AOB and $A'OB'$ before and after the avalanche, respectively. Since it is assumed that there is no diffusion inside the wedge, each grain retains its location relatively to the local frame of reference $\mathbf{e}_1$, $\mathbf{e}_2$ (the point A is mapped onto A', B onto B', etc.). Therefore, the avalanche motion of the wedge AOB can be described as its rotation by 180° and deformation. Since the area of the triangle is assumed to be constant, one of its sides is stretched while the other is squeezed:

$$\frac{|AO|}{|A'O|} = \frac{|B'O|}{|BO|} \approx 1 + \lambda \cdot (\Delta\theta)\,,$$

where $|\lambda|$ is the rate of the material deformation. The degree of mixing can be measured in terms of the number of contacts between particles of different types per unit volume (*i.e.* the length of the intermaterial boundary in the two-dimensional case). As was demonstrated by Ottino *et al.* [12], the rate of material deformation provides the upper limit to the generated intermaterial length and can be used for estimating the macroscopic mixing capacity of the container.

Consider a flow of granular material more in detail. Since the rotating mixer is convex and the bulk density is constant, the location of the free surface, *i.e.* $z = h(\theta)$, can be easily calculated from mass conservation as follows. The transformation from polar to Cartesian coordinates at each moment of time θ (see fig. 1) reads

$$x = r(\phi)\cos(\phi + \theta), \qquad z = r(\phi)\sin(\phi + \theta)\,.$$

Note that the coordinates x, z are determined in the laboratory frame while the polar coordinates r, ϕ are determined in a frame of reference attached to the rotating drum. Thus the distance from the free surface to the bottom of the container is given by

$$\xi = [h(\theta) - r(\theta)\sin(\phi + \theta)]_+\,,$$

where $[\xi]_+ = \max(\xi, 0)$. The two-dimensional volume occupied by the granular material placed in the mixer is $V = \int \xi \mathrm{d}x$. Therefore the height of the free surface $h(\theta)$ is a solution of the following equation:

$$\int_0^{2\pi} [h(\theta) - r(\theta)\sin(\phi + \theta)]_+ d\big(r(\phi)\cos(\phi + \theta)\big) = V\,. \tag{1}$$

Let, at time θ, the free surface be located at $z = h(\theta)$ and have length $2L(\theta)$. At time $\theta + \Delta\theta$ consider two lines BB' with the equation

$$z = h(\theta) + (\Delta\theta)\frac{\mathrm{d}h(\theta)}{\mathrm{d}\theta} + \frac{1}{2}(\Delta\theta)^2\frac{\mathrm{d}^2h(\theta)}{\mathrm{d}\theta^2} + \ldots \tag{2}$$

and AA' which rotates with the container. The equation for the line AA' can be calculated from the following kinematic equation of the free surface $z = Z(\theta, x)$:

$$\frac{\partial Z(\theta,x)}{\partial\theta} + V_x\frac{\partial Z(\theta,x)}{\partial x} - V_z = 0,$$

with the initial condition $Z(\theta, x) = h(\theta)$. Hereafter, Z denotes a Lagrangian coordinate of the free surface, while z denotes the Eulerian coordinate. Since the velocity field in the container is given by $V_z = x\dot{\theta}$ and $V_x = -z\dot{\theta}$, the equation for the free surface reduces to

$$\frac{\partial Z(\theta,x)}{\partial\theta} = Z(\theta,x)\frac{\partial Z(\theta,x)}{\partial x} + x\,.$$

The solution of the latter equation reads

$$Z(\theta + \Delta\theta, x) = h(\theta) + x(\Delta\theta) + \frac{(\Delta\theta)^2}{2}h(\theta) + \ldots\,. \tag{3}$$

Equating (2) and (3) we find that the intersection of lines AA' and BB' is located at

$$x_O = \frac{\mathrm{d}h(\theta)}{\mathrm{d}\theta} + \frac{\Delta\theta}{2}\left(\frac{\mathrm{d}^2h(\theta)}{\mathrm{d}\theta^2} - h(\theta)\right)\,.$$

Consider what happens when $\Delta\theta \to 0$. In the latter case $|BO'| \to |BO|$ and $x_O \to \mathrm{d}h(\theta)/\mathrm{d}\theta$, *i.e.* at each moment θ the center of the free surface C is located at $x_C = \mathrm{d}h(\theta)/\mathrm{d}\theta$. Therefore, at time $\theta + \Delta\theta$ the center of the free surface is located at $x_C = \mathrm{d}h(\theta)/\mathrm{d}\theta + \mathrm{d}^2h(\theta)/\mathrm{d}\theta^2(\Delta\theta)$

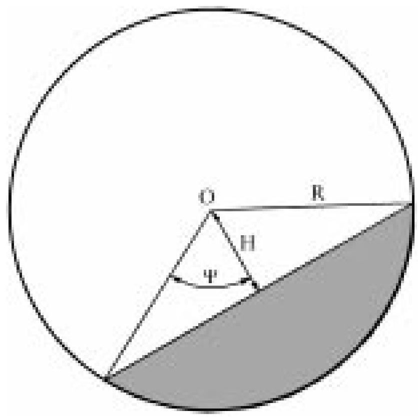

Fig. 3. – Geometrical parameters of a circular mixer.

and

$$|AO| = L + (x_O - x_C) = L + \frac{\Delta\theta}{2}\left(h + \frac{\mathrm{d}^2 h}{\mathrm{d}\theta^2}\right),$$

$$|A'O| = L - (x_O - x_C) = L - \frac{\Delta\theta}{2}\left(h + \frac{\mathrm{d}^2 h}{\mathrm{d}\theta^2}\right).$$

From the above formulas we immediately obtain an expression for the stretching rate:

$$\lambda = \lim_{\Delta\theta\to 0}\left(\frac{1}{\Delta\theta}\frac{|AO|}{|A'O|} - 1\right) = \frac{1}{L}\left(h + \frac{\mathrm{d}^2 h}{\mathrm{d}\theta^2}\right). \tag{4}$$

Equation (4) shows that the granular material experiences the minimum stretching rate in a half-filled circular drum. Indeed, in the latter case $h \propto \sin(\theta)$ and $\lambda = 0$, each particle moves along the half-circle closed trajectory with rotation period $\Theta = \pi$ without mixing.

When a container with volume V_0 is filled with granular material occupying a volume V, the mixing efficiency μ can be defined as the product of the mean deformation per unit volume of the avalanching material and the filling level of the granular material in a mixer $f = \frac{V}{V_0}$:

$$\mu = f\frac{\displaystyle\int_0^{2\pi} |\lambda|\frac{L^2}{2}\mathrm{d}\theta}{V}.$$

Using eq. (4), after the same algebra the latter formula can be written as follows:

$$\mu = \frac{1}{2V_0}\int_0^{2\pi}\left|h + \frac{\mathrm{d}^2 h}{\mathrm{d}\theta^2}\right| L\mathrm{d}\theta. \tag{5}$$

Consider the circular mixer with radius R (see fig. 3). Then

$$\mu = \frac{1}{2}|\sin 2\Psi|, \quad f = \frac{1}{\pi}\left(\Psi - \frac{1}{2}\sin 2\pi\Psi\right) \quad \text{and} \quad h(\theta) = H.$$

Thus, the maximum stretching rate $\mu_{\max} = 0.5$ corresponds to $\Psi = \frac{\pi}{4}$, *i.e.* $f_{\mathrm{opt}} = \frac{1}{4} - \frac{1}{2\pi} \approx 0.09$, $H_{\mathrm{opt}} = \frac{R}{\sqrt{2}} \approx 0.71\,R$. The latter result is consistent with that obtained in [13] for the

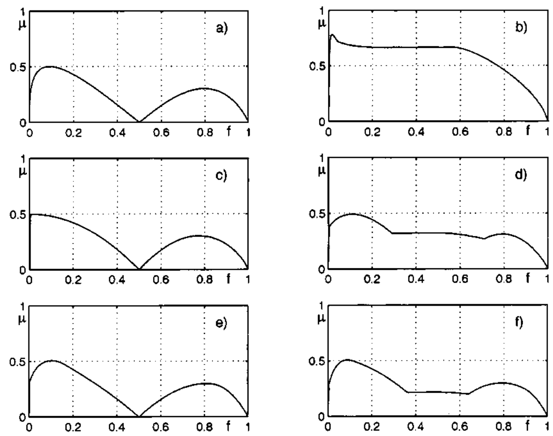

Fig. 4. – Mixing efficiency index μ *vs.* filling level f: a) circular drum; b)–f) triangle, square, pentagon, hexagon and heptagon, respectively.

case of a low particle diffusivity, namely, $f_{\rm opt} \approx 0.11$, $H_{\rm opt} \approx 0.65$. Note that the range of the typical filling levels applied in industry varies from 8% to 15%.

When the free surface of the granular material reaches the center of a mixer, a core of material which does not get involved in the flow is formed. This region only rotates with the container and reduces the effective amount of a mixed material. Thus, the mixing efficiency is decreased by the factor $(V - V_{\rm core})/V$ and eq. (5) becomes

$$\mu = \frac{V - V_{\rm core}}{2V_0 V} \int_0^{2\pi} \left| h + \frac{\mathrm{d}^2 h}{\mathrm{d}\theta^2} \right| L \mathrm{d}\theta \,. \tag{6}$$

The dependence of the mixing efficiency index μ for a circular container *vs.* filling level is plotted in fig. 4a). This dependence has two maxima: one before and another after the core formation. We examined also the mixing efficiency of the equilateral polygons (triangle, square, pentagon, hexagon, heptagon). In the numerical solution eq. (1) was solved for each value of θ by a bisection method. The obtained solution was used to calculate the mixing efficiency from eq. (6). The results of these computations are presented in fig. 4b)-4f). Inspection of these figures shows that all the container shapes, except for the triangle, exhibit a similar behavior to the mixing index μ. The maximum value of μ is 0.5, the minimum μ corresponds to a half-filled mixer. When a container is half-filled ($f = 0.5$) the polygons with an even number of sides have zero mixing efficiency, while for an odd number of sides the dependence $\mu(f)$ has a plateau with a relatively low values of μ which is formed in the central part of the plot. Thus, these mixers have no advantage over the circular mixer used in industry. The triangular mixer has the best mixing index among the shapes of the mixers examined in this study. The maximum value of the mixing index $\mu = 0.77$ is attained at a filling level $f = 0.02$. After that there is a long interval in the triangular mixer with a nearly constant mixing index $\mu \approx 0.7$ for $f \in [0.1, 0.6]$, *i.e.* even for high values of the filling level the mixing rate in a triangular container is higher than the maximum rate of mixing for the circular mixer. Therefore, the triangular shape of the container provides higher throughput than other mixers. Notably, the mixing efficiency obtained using the model with a perfect mixing inside the avalanching granular wedge and that measured in the experiments with small containers [7] also indicates that the triangular shape is the best shape for a mixer.

In summary, we have proposed a kinematic model for the mixing of granular material in a slowly rotated container in the case of low particles diffusivity. The powder flow in a circular container is analyzed analytically. The efficiency of the mixers with different shapes is determined as a function of filling level.

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