

Analytical reconsideration of the von Neumann paradox in the reflection of a shock wave over a wedge

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The reflection of weak shock waves has been reconsidered analytically using shock polars. Based on the boundary condition across the slipstream, the solutions of the three-shock theory (3ST) were classified as “standard-3ST solutions” and “nonstandard-3ST solutions.” It was shown that there are two situations in the nonstandard case: A situation whereby the 3ST provides solutions of which at least one is physical and a situation when the 3ST provides a solution which is not physical, and hence a reflection having a three-shock confluence is not possible. In addition, it is shown that there are initial conditions for which the 3ST does not provide any solution. In these situations, a four-wave theory, which is also presented in this study, replaces the 3ST. It is shown that four different wave configurations can exist in the weak shock wave reflection domain, a Mach reflection, a von Neumann reflection, a ?R (this reflection is not named yet!), and a modified Guderley reflection (GR). Recall that the wave configuration that was hypothesized by Guderley [“Considerations of the structure of mixed subsonic-supersonic flow patterns,” Air Materiel Command Technical Report No. F-TR-2168-ND, ATI No. 22780, GS-AAF-Wright Field No. 39, U.S. Wright-Patterson Air Force Base, Dayton, OH (October 1947); *Theorie Schallnaher Strömungen* (Springer-Verlag, Berlin, 1957)] and later termed Guderley reflection did not include a slipstream (see Fig. 7). Our numerical study revealed that the wave structure proposed by Guderley must be complemented by a slipstream (see Fig. 4) in order to be relevant for explaining the von Neumann paradox. Hereafter, for simplicity, this modified GR wave configuration will be also termed Guderley reflection. The domains and transition boundaries between these four types of reflection are elucidated. © 2008 American Institute of Physics. [DOI: 10.1063/1.2896286]

INTRODUCTION

The phenomenon of shock wave reflection was first reported by Mach.¹ In his experiments, he discovered two types of shock wave reflection configurations: Regular reflection (RR) and irregular reflection (IR) with a shock branching point (triple point). This wave configuration was later named after him and is known nowadays as the Mach reflection (MR). In order to describe the shock wave configuration near the triple point, von Neumann^{2,3} suggested a simple three-shock theory (3ST) that was based on the conservation equations across the three oblique shock waves and appropriate boundary conditions across the slipstream.

Theoretical analysis⁴ for the reflection of pseudostationary shock wave from a wedge revealed some contradictions in the von Neumann theory. In particular, the theory yielded nonphysical configurations with an incoming reflected shock wave and failed to recover the limit that corresponds to a degenerated case with a wedge angle $\theta_w \rightarrow 0$. A theoretically consistent solution instead of the nonphysical branch in the von Neumann theory was first suggested by Guderley.^{5,6} Using approximation of potential (isentropic) flow for weak shock waves, Guderley concluded that a supersonic patch exists behind the triple point. The isentropic solution of Guderley could not eliminate the contradiction in von Neumann's theory since it did not include a tangential discontinuity (see

Fig. 7). However, it contained two important elements: A reflected shock wave that was directed toward the incoming flow and a supersonic patch behind the triple point.

Discrepancies between von Neumann's 3ST and experiments were first detected by White.^{7,8} Apart from quantitative discrepancies, in the experiments for weak shock waves with $M_S < 1.5$, he recorded MR configurations in a parameter domain where the 3ST did not allow such a solution. Numerous experiments that were later reported in Refs. 9–13 confirmed these discrepancies, which were named *the von Neumann paradox*. Measurements of the shock front angles in the experiments persistently showed that the reflected shock waves were outcoming. These findings did not comply neither with predictions based on von Neumann's 3ST (where it admits a solution) nor with Guderley's solution. The latter, probably, was the decisive reason for abandoning Guderley's solution. Subsequent efforts were directed toward searching theoretical models that could explain the experimental results, and in many suggested models, it was assumed *a priori* that the reflected shock wave cannot be outcoming (i.e., directed toward the incoming flow).

The efforts to resolve theoretically the von Neumann paradox were reduced mainly to modifications or revisions of the 3ST. One of the commonly used approaches to resolve the von Neumann paradox was to account for viscous effects.^{14,15} It should be noted that although viscous effects

undoubtedly exist in the experiments, their influence diminishes as the distance between the shock wave and a leading edge of the reflecting wedge increases. These effects should have been revealed through the violation of self-similarity of the flow parameters that was not observed in the experiments.

One of the most popular attempts to revise the theory in the framework of the ideal gas model was the adoption of the so-called nontraditional scheme which allowed nonparallel flow velocities on both sides of the tangential discontinuity.^{9,12,16–19} The result of abandoning the condition for parallel flow velocities on both sides of the contact discontinuity was the formation of a narrow expanding sector with undetermined pressure near the triple point. In order to determine this pressure, Shindyapin¹⁹ employed the principle of the maximum deflection of the flow after passing through a reflected shock wave. Note that like Guderley's solution, this principle implied that the reflected shock wave is directed against the flow and deflects the streamlines toward the wall. Clearly, the nontraditional scheme is quite artificial, and its weak point is that it does not suggest reasonable explanations why this expanding sector with undetermined pressure has not been observed experimentally. In addition to the nontraditional scheme, Ref. 12 relaxed the requirement of equal pressures on both sides of the tangential discontinuity and postulated a linear relationship of the pressure behind the reflected shock wave and the triple-point trajectory angle. For calibrating this linear dependence, they used the degenerated solution with $\theta_w=0$. In spite of the absence of any theoretical foundation, the authors managed to obtain good agreement with experimental results in a wide range of parameters. Therefore, the results obtained in Ref. 12 implied that the correct physical theory that could resolve the paradox must contain a continuous transition to the limiting acoustic solution with $\theta_w=0$.

The first serious attempt to simulate numerically shock wave reflection under the conditions of the von Neumann paradox was undertaken by Colella and Henderson¹¹ using a second-order accurate scheme. Based on their calculations, which were performed on a fixed orthogonal grid, they hypothesized that the reflected shock wave degenerated into a continuous compression wave near the triple point. However, Olim and Dewey¹² showed that experiments comply with this hypothesis only when $M_S < 1.05$ or for wedge angle $\theta_w < 10^\circ$. Under different conditions, the reflected wave propagates with supersonic velocity, and it is a shock wave with finite amplitude. This suggests that the smearing of the shock wave front in the numerical calculations of Ref. 12 was a numerical artifact of their shock capturing scheme, and that numerical flow modeling near the triple point requires a considerably higher resolution.

Calculations performed by Vasilev^{20,21} using a numerical scheme with a very high resolution and new front tracking techniques confirmed the principal points of Guderley's solution. The numerical calculations revealed the existence of an expansion fan emerging from the triple point with a supersonic patch behind it. These results allowed formulating a *four-wave theory* (4WT), which completely resolved the von Neumann paradox and included the degenerated case with

$\theta_w=0$. The governing equations of the 4WT are presented in the following. It was shown that the 4WT complies fairly well with the experimental results of Ref. 12. Vasilev and Kraiko²² performed similar calculations on adaptive grids and, using extrapolation, they showed that the numerical results agreed with a generalized three-wave theory/4WT in the whole range of the wedge angles θ_w . The numerical results and theoretical analysis conducted by Vasilev²³ showed that for weak shock waves, a very small logarithmic singularity with very large (infinite) gradients of the flow variables was formed near the triple point. The curvature of the reflected shock wave at the triple point also approached infinity. Because of the large curvature of the reflected shock wave, the subsonic flow behind it converges and becomes supersonic. Owing to the small sizes of both the singularity and of the supersonic zone, they could not be detected in experiments with affordable resolution, and probably, this was an additional reason for the emergence of the von Neumann paradox.

Since Guderley did not perform an analysis of the curvature of the reflected shock wave, he assumed that the curvature could be finite, and in his solution, instead of a centered-expansion fan at the triple point, he used a continuous expansion wave with characteristics emerging from the Mach stem. However, the numerical solution reveals the existence of configurations where the flow behind the Mach stem is subsonic, in general, with a supersonic flow region, which is in contact with the shock waves only at the triple point. Such a configuration is essentially different from that described by Guderley's solution since it cannot be described using his potential isentropic flow model.

Tesdall and Hunter²⁴ conducted numerical calculations using a simplified model based on the two-dimensional Burgers equation. Recently, Tesdall *et al.*²⁵ performed similar calculations using the model of the nonlinear wave system. They achieved a high numerical resolution that was sufficient for the analysis of the fine structure of the flow field near the triple point under the conditions of the von Neumann paradox. They discovered a complex cascade of alternating expansion fans and weak shock waves that connects the supersonic flow regions. Guderley⁶ already noted, when describing his solution, that in the framework of the isentropic model (i.e., without a contact discontinuity), there could be a weak shock wave at the edge of the supersonic flow patch, and therefore, the structure of the flow field could be quite complex. Accounting for the fact that both models are isentropic, it is conceivable to suggest that the solution obtained in Ref. 24 coincides with that proposed by Guderley. It must be noted that, most probably, in the Euler model, the cascade solution does not exist. Isentropic models are close to the Euler model either when the initial shock wave is very weak or when the wedge angle is very small. In Refs. 24 and 25, both these conditions are not satisfied, e.g., the wedge angle is larger than 30° . Under this condition, the wave configuration obtained using the Euler model has a pronounced contact discontinuity (slipstream) behind the triple point. The Mach number below the slipstream is considerably smaller than the Mach number above the slipstream. Therefore, the size of the supersonic patch below the slipstream is substan-

tially smaller than the size of the supersonic patch above the slipstream. Consequently, the characteristic lines cannot intersect in the supersonic zone below the slipstream and a new (closing) shock wave is not formed. Probably, the complex multipatch structure exists in the range of very weak shock waves. Similar flow configurations were observed by Henderson *et al.*²⁶ in solving a slightly different problem. A definite answer to this question requires a detailed and accurate investigation of very weak shock wave domain. In addition, it is necessary to take into account that for very weak shock waves, the time required for formation of a self-similar flow increases. It is conceivable that the cascade structure is only a transient flow pattern, which is formed during transition to the self-similar flow, both when investigated numerically and experimentally.

The final closure to the 50 year history of the von Neumann paradox was achieved by Skews and Ashworth.²⁷ The above-mentioned expansion fan was detected in their unique experiments using a large-scale installation. At the same time, their conclusion about the existence of the sequence of closing shock waves and cascade based on postprocessing of the photographs requires further verification.

However, there are still several unsolved problems associated with the reflection of weak shock waves. It was noted above that the integral feature of weak shock wave reflection is the existence of a singularity with infinite gradients of flow field variables at the triple point. The formation of this singularity could be qualified as the boundary between MR and von Neumann reflection (vNR). Calculations performed by Vasilev^{20,21} showed that this singularity is formed earlier than as was considered before, i.e., for $\phi_2 < 90^\circ$. Hence, this boundary must be determined more exactly. Although the infinite value of the curvature of the reflected shock wave at the triple point in the Guderley reflection (GR) region is established quite reliably, the problem of the curvatures of the contact surface and the Mach stem remains open. Calculations^{20,21} showed that the curvature of the Mach stem in GR is finite. Unfortunately, the calculations did not allow determining reliably the curvature of the contact surface. This aspect of the problem is very important since it is directly associated with the propagation character of the compression waves in the supersonic regions and, consequently, with the existence or nonexistence of the closing shock wave. Note that Guderley noted that a high curvature of the boundary of the supersonic region could be the reason for the nonexistence of a closing shock wave.

THEORETICAL BACKGROUND

In the following, the governing equations of the 3ST and the 4WT are presented.

The three-shock theory

The analytical approach for describing the MR wave configuration, the 3ST, was pioneered by von Neumann.^{2,3} It makes use of the inviscid conservation equations across an oblique shock wave, together with appropriate boundary conditions.

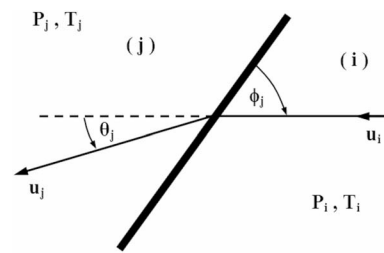


FIG. 1. Definition of parameters across an oblique shock wave.

Consider Fig. 1 where a supersonic flow flowing from right to left toward an oblique shock wave is shown. The flow states ahead and behind it are *i* and *j*, respectively. The angle of incidence between the incoming flow and the oblique shock wave is ϕ_j . While passing through the oblique shock wave, from state *i* to state *j*, the flow is deflected by an angle θ_j . The conservation equations across an oblique shock wave, relating states *i* and *j* for a steady inviscid flow of a perfect gas, read

$$\rho_i u_i \sin \phi_j = \rho_j u_j \sin(\phi_j - \theta_j) \quad (\text{conservation of mass}), \quad (1a)$$

$$p_i + \rho_i u_i^2 \sin^2 \phi_j = p_j + \rho_j u_j^2 \sin^2(\phi_j - \theta_j) \quad (\text{conservation of normal momentum}), \quad (1b)$$

$$\rho_i \tan \phi_j = \rho_j \tan(\phi_j - \theta_j) \quad (\text{conservation of tangential momentum}), \quad (1c)$$

$$\frac{\gamma}{\gamma - 1} \frac{p_i}{\rho_i} + \frac{1}{2} u_i^2 \sin^2 \phi_j = \frac{\gamma}{\gamma - 1} \frac{p_j}{\rho_j} + \frac{1}{2} u_j^2 \sin^2(\phi_j - \theta_j) \quad (\text{conservation of energy}) \quad (1d)$$

Here, u is the flow velocity in a frame of reference attached to the oblique shock wave, ρ and p are the flow density and static pressure, respectively, and $\gamma = C_p/C_v$ is the specific heat capacity ratio. Consequently, the above set of four equations contains eight parameters: p_i , p_j , ρ_i , ρ_j , u_i , u_j , ϕ_j , and θ_j .

The wave configuration of a MR is shown schematically in Fig. 2. It consists of four discontinuities: Three shock waves (the incident shock wave *i*, the reflected shock wave *r*,

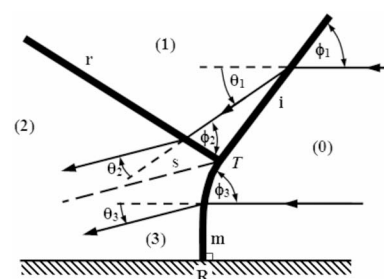


FIG. 2. Schematic illustration of the wave configuration of a MR.

and the Mach stem m) and one slipstream s . These four discontinuities meet at a single point, the triple point T , which is located above the reflecting surface. The Mach stem is slightly curved along its entire length. At its foot R , it is perpendicular to the reflecting surface.

The conservation equations across the three shock waves that meet at the triple point could be obtained from the just-mentioned oblique shock wave equations by setting $i=0$ and $j=1$ for the incident shock, $i=1$ and $j=2$ for the reflected shock, and $i=0$ and $j=3$ for the Mach stem.

In addition to these 12 conservation equations, there are also two boundary conditions, which arise from the fact that states 2 and 3 of the MR configuration (Fig. 2) are separated by a contact surface across which the pressure remains constant,

$$p_2 = p_3. \quad (2)$$

Furthermore, if the flow is assumed to be inviscid and if the contact surface is assumed to be infinitely thin (i.e., a slipstream), then the streamlines on its both sides are parallel, i.e.,

$$\theta_1 \pm \theta_2 = \theta_3. \quad (3)$$

Based on Eq. (3), we divide the 3ST into two types:

- A “standard” 3ST for which

$$\theta_1 - \theta_2 = \theta_3. \quad (4a)$$

- A “nonstandard” 3ST for which

$$\theta_1 + \theta_2 = \theta_3. \quad (4b)$$

In general, the 3ST (either the standard or the nonstandard) comprises 14 governing equations that contain 18 parameters, namely, $p_0, p_1, p_2, p_3, \rho_0, \rho_1, \rho_2, \rho_3, u_0, u_1, u_2, u_3, \phi_1, \phi_2, \phi_3, \theta_1, \theta_2$, and θ_3 .

As will be shown subsequently, a solution of the standard 3ST yields a MR (it should be noted here that most of, if not all, the textbooks present the standard 3ST when they describe the boundary conditions across the slipstream), while a solution of the nonstandard 3ST includes two cases: A case in which the solution is physical and a case in which the solution is not physical. While in the former case the resulting reflection is a vNR, in the latter case, it is a new type of reflection, ?R (this reflection is not named yet!), which in fact is an intermediate wave configuration between the vNR and the earlier mentioned modified GR. In the ?R case, the flow between the Mach stem and contact surface is subsonic and has a logarithmic singularity with infinite gradients near the triple point. A similar situation also occurs in the flow patch between the reflected shock wave and the expansion fan.

The just-mentioned three different wave configurations, vNR, ?R, and GR, are shown in Fig. 3. In the vNR, all the flow regions behind the reflected shock wave and the Mach stem are subsonic. Hence, the triple point is embedded in a subsonic flow with a logarithmic singularity (except for the incoming flow behind the reflected shock wave). In the ?R case, there is one supersonic patch (white area) that extends from the slipstream toward the reflected shock wave with a

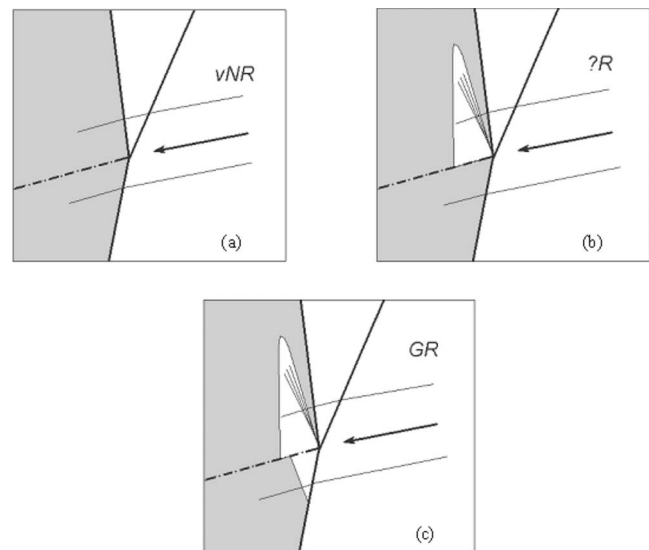


FIG. 3. Schematic illustration of the wave configurations of a vNR, the newly presented reflection ?R, and the GR. Gray denotes subsonic flow.

Prandtl–Meyer expansion fan inside it. In the GR, there are two supersonic patches. The first one is similar to that in the ?R case, which extends from the slipstream toward the reflected shock wave with a Prandtl–Meyer expansion fan inside it, and the second one is that located between the slipstream and the Mach stem. Consequently, while in the ?R case there are two subsonic regions near the triple point, one behind the reflected shock wave and one behind the Mach stem, in the GR, there is only one subsonic region near the triple point, behind the reflected shock wave. Thus, the vNR, ?R, and GR are characterized by different numbers of subsonic regions with logarithmic singularity near the triple point. It should be noted that the size of the subsonic regions is significant. The larger the subsonic region is, the smaller the characteristic size of the singularity is. Therefore, in vNR, the size of the singularity is smaller than in ?R, which is smaller than in GR.

It should be also noted here that the vNR wave configuration was hypothesized by Colella and Henderson¹¹ as a possible resolution of the von Neumann paradox. Note that the von Neumann paradox refers to situations in which wave configurations that look like MR appear in a parameter domain for which the 3ST does not have any solution. However, since as mentioned above, the vNR appears in a parameter domain where there is a solution of the nonstandard 3ST, this hypothesis cannot resolve the von Neumann paradox! It will be shown subsequently that the modified GR most likely resolves the von Neumann paradox.

The four-wave theory

When the solution of the 3ST is not physical, the resulting wave configurations, ?R and GR, are described by 4WT. The wave configuration and the definitions of the flow regions under these circumstances are shown in Fig. 4. The wave configuration consists of the shock waves (i, r , and m),

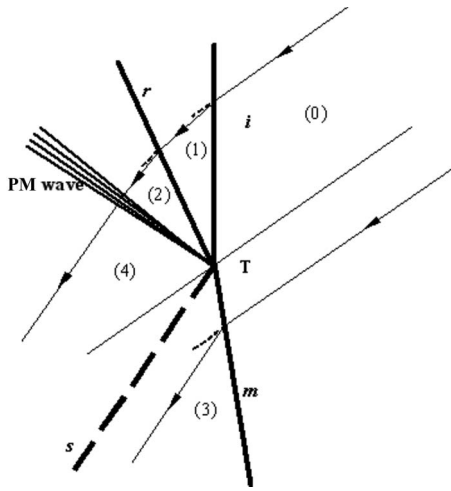


FIG. 4. Schematic illustration of a four-wave configuration and definition of parameters.

one Prandtl–Meyer expansion wave, and one slipstream across which the pressures and flow directions are identical.

The equations across the incident i , reflected r , and Mach m , shocks are obtained from Eqs. (1a)–(1d) with the appropriate values of i and j .

The total pressures on both sides of the Prandtl–Meyer expansion fan are the same; hence

$$p_2 \left(1 + \frac{\gamma-1}{2} M_2^2 \right)^{\gamma/(\gamma-1)} = p_4 \left(1 + \frac{\gamma-1}{2} M_4^2 \right)^{\gamma/(\gamma-1)}, \quad (5)$$

where M_2 and M_4 are the flow Mach numbers ahead and behind the expansion fan, respectively (see Fig. 4). The flow deflection across the expansion fan is

$$\theta_4 = \nu(M_4) - \nu(M_2), \quad (6)$$

where $\nu(M)$, the Prandtl–Meyer function, is given by the following expression:

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma+1}{\gamma-1} (M^2 - 1)} - \tan^{-1} \sqrt{M^2 - 1}.$$

For closing the 4WT, we require the following:

$$M_2 = 1. \quad (7)$$

The boundary conditions across the slipstream are

$$\theta_1 + \theta_2 + \theta_4 = \theta_3, \quad (8a)$$

$$p_3 = p_4. \quad (8b)$$

The $\nu\text{NR} \leftrightarrow ?\text{R}$ transition in the 4WT corresponds to $p_4 = p_2$. This is equivalent to the condition $M_2 = 1$ in the 3ST. It is a common point of the 3ST and the 4WT (i.e., $3\text{ST} \leftrightarrow 4\text{WT}$). The $?\text{R} \leftrightarrow \text{GR}$ transition in the 4WT corresponds to $M_3 = 1$.

As will be shown subsequently, the boundaries between the domains of existence of solutions of the 3ST, including nonphysical solutions (line 5 in Fig. 10), and of the 4WT

(line 4 in Fig. 10) are different, though both correspond to the same condition, $p_1 = p_2$, i.e., zero amplitude of the reflected shock wave.

SHOCK-POLAR PRESENTATION

Kawamura and Saito²⁸ were the first to suggest that owing to the fact that the boundary conditions of the 3ST are expressed in terms of the flow static pressures [Eq. (2)] and the flow-deflection angles [Eq. (3)], the use of (p, θ) polars could assist in better understanding the shock wave reflection phenomenon.

The graphical presentation of the relationship between the pressure p_j that is obtained behind an oblique shock wave and the angle θ_j by which the flow is deflected while passing through the oblique shock wave (see Fig. 1) for a fixed flow Mach number M_i and different angles of incidence ϕ_j is known as a pressure-deflection shock polar. A detailed description of shock polars and their application to shock wave reflections can be found from Ben-Dor.²⁹

Shock-polar presentation of the flow field in the vicinity of the triple point

It should be noted that shock polars and the equations they represent are based on the assumption of the wave being plane, and the flow immediately behind it therefore being uniform. As stated before, both the reflected shock and Mach stem are curved, although in all numerical and experimental studies done to date, a plane wave would appear to be a reasonable assumption. The important point, however, is to recognize that it is the wave curvature which results in the acceleration of the flow from subsonic to sonic conditions behind the reflected wave for both GR and ?R, as shown in Fig. 3. The effects of this assumption are of much less concern when determining transition boundaries, as will be discussed later. Three interesting $(I\text{-}R)$ -polar combinations, corresponding to detachment, are shown in Figs. 5(a)–5(c). Since the reflected shock wave could be directed either away [as shown in Fig. 6(a)] or toward [as shown in Fig. 6(b)] the incoming flow, both the left and right branches of the R polar are drawn. Figure 5(a) presents an $(I\text{-}R)$ -polar combination for which the net deflection of the flows in state 2, with respect to the triple point, is smaller than that in state 1. Hence, the flow originating from state 0 at a point above the triple-point trajectory is first deflected by the incident shock wave toward the reflecting wedge surface, and then it is deflected by the reflected shock wave away from the reflecting wedge surface to result in $\theta_2^T = \theta_3^T < \theta_1^T$ (here, θ_i^T is measured with respect to the direction of the incoming flow when the frame of reference is attached to the triple point T). This implies that $\theta_1 - \theta_2 = \theta_3$ [Eq. (4a)], and hence the $(I\text{-}R)$ -polar combination represents a standard solution of the 3ST. The $(I\text{-}R)$ -polar combination shown in Fig. 5(b) illustrates a different solution. Here, the flow that is deflected toward the reflecting wedge surface while passing through the incident shock wave is not deflected away from the reflecting wedge surface [as in Fig. 5(a)] when it passes through the reflected shock wave but further deflected toward the reflecting wedge surface to result in $\theta_2^T = \theta_3^T > \theta_1^T$. This implies that $\theta_1 + \theta_2 = \theta_3$

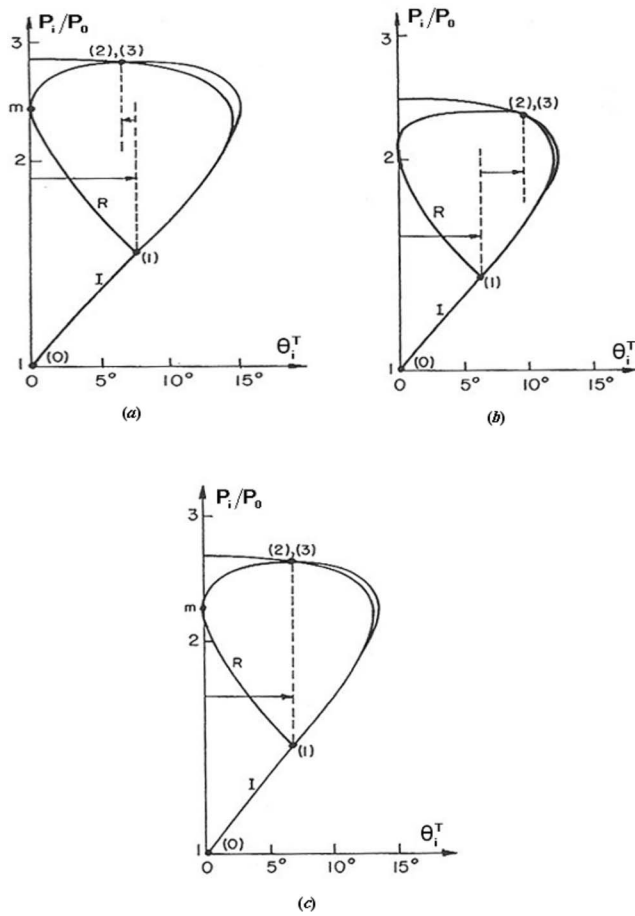


FIG. 5. (I - R)-polar presentation of a possible MR solution for which (a) $\theta_2^T = \theta_3^T < \theta_1^T$ ($M_0=1.6$, $\phi_1=47.88^\circ$, and $\gamma=1.4$); (b) $\theta_2^T = \theta_3^T > \theta_1^T$ ($M_0=1.5$, $\phi_1=49.67^\circ$, and $\gamma=1.4$); and (c) $\theta_2^T = \theta_3^T = \theta_1^T$ ($M_0=1.55$, $\phi_1=41.50^\circ$, and $\gamma=1.4$).

[Eq. (4b)], and hence this (I - R)-polar combination represents a nonstandard solution of the 3ST. Note that in the wave configuration corresponding to this case, the reflected shock wave is directed toward the incoming flow. The limiting (I - R)-polar combination between the standard and nonstandard solutions is shown in Fig. 5(c), which indicates that the flow passing through the reflected shock wave is not deflected at all, i.e., $\theta_2=0$, and hence $\theta_2^T = \theta_3^T = \theta_1^T$. The boundary condition of the 3ST for this case is $\theta_1 = \theta_3$. This implies that the reflected shock wave is normal to the incoming flow in state 1, i.e., $\phi_2=0$. This condition corresponds to the MR $\leftrightarrow$ vNR transition.¹¹ The three-wave configurations that correspond to the three (I - R)-polar combinations that are presented in Figs. 5(a)–5(c) are shown in Figs. 6(a)–6(c). The earlier remarks regarding the orientations of the reflected shock waves are evident from inspecting these figures.

Colella and Henderson¹¹ concluded that

- When the solution of the governing equations of the 3ST results in a standard solution for which $\theta_1 - \theta_2 = \theta_3$, the reflection is a MR, and
- when the solution of the governing equations of the 3ST results in a nonstandard solution for which $\theta_1 + \theta_2 = \theta_3$, the reflection is a vNR.

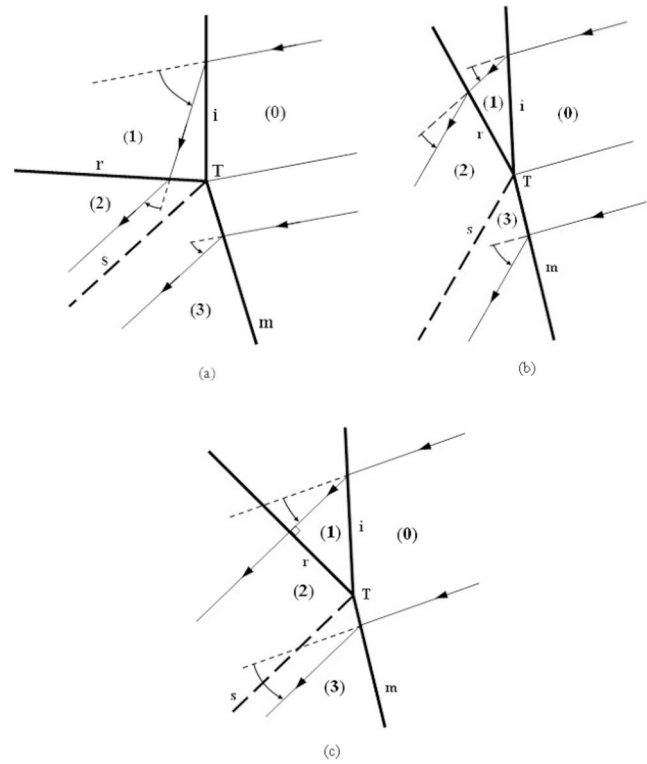


FIG. 6. The wave configurations of the three possible solutions of the three-shock theory whose graphical solutions are shown in Figs. 5(a)–5(c), respectively.

The latter statement contradicts the concept forwarded by Colella and Henderson¹¹ that the vNR resolves the von Neumann paradox. The vNR occurs not when the 3ST does not have a solution (i.e., when the I and R polars do not intersect) but when the 3ST provides a nonstandard solution, i.e., a solution for which $\theta_1 + \theta_2 = \theta_3$ rather than $\theta_1 - \theta_2 = \theta_3$, as is the case of a standard solution, which results in a MR.

As will be shown subsequently, while the first statement is correct, the second one is only partially correct since there are cases in which the 3ST does have a nonstandard solution but the solution is not physical. Hence, the second statement should be rephrased to read as follows:

- When the solution of the governing equations of the 3ST results in a nonstandard solution for which $\theta_1 + \theta_2 = \theta_3$, the reflection is a vNR *if the solution is physical*.

Now that the picture is clear when the 3ST provides standard and nonstandard solutions that are physical, there are still two unresolved situations:

- (1) The 3ST has a nonstandard solution, which is not physical. In this case, the flow Mach number behind the reflected shock wave, M_2 , is larger than 1. Consequently, the reflected shock wave is an incoming wave with respect to the triple point. The latter situation is not physical.
- (2) There are experimental situations in which the recorded reflections resemble MR wave configurations in a do-

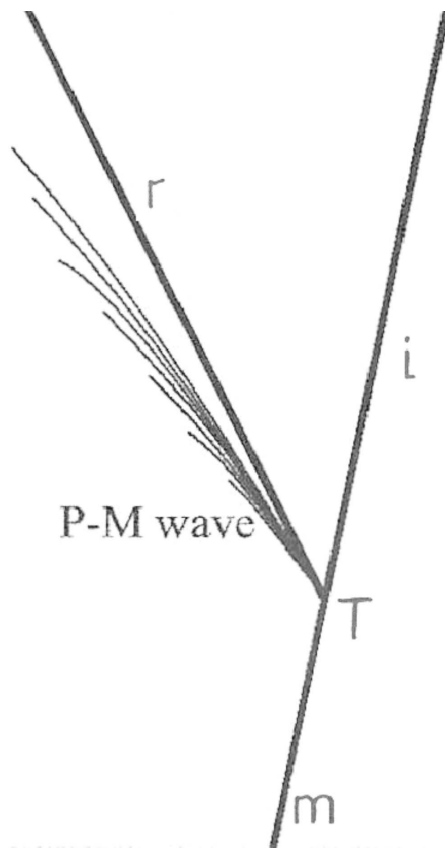


FIG. 7. Schematic illustration of a GR.

main where the 3ST does not have any solution (i.e., the I and R polars do not intersect at all). This is, in fact, the essence of the von Neumann paradox.

As it turns out, a possible resolution of this paradox was forwarded over 60 years ago by Guderley,⁵ who claimed that, in addition to the three shock waves that meet at a single point (the triple point), there are cases in which a tiny Prandtl–Meyer expansion fan complements the wave configuration. Hence, rather than a three shock wave confluence, there is a four-wave confluence (three shock waves and one expansion wave). For this reason, it should not be surprising that the 3ST failed to describe this wave configuration since it never intended to model a four-wave confluence. Following Skews and Ashworth,²⁷ this reflection [see Figs. 3(c) and 7] has been named after Guderley and is called *Guderley reflection*.

However, as it turned out in the course of the present study, the foregoing analysis still does not provide a full picture of the phenomenon. As will be shown subsequently, there is an intermediate domain, between the vNR and the GR domains, where none of these two wave configurations is possible and there is an additional wave configuration in this intermediate domain, which has not been reported yet.

The existence of this intermediate domain is shown in the following by means of the evolution of the $(I-R)$ -polar combinations as the complementary wedge angle, $\theta_w^C = 90^\circ - \phi_1$, is decreased from an initial value of 41° , for which the

reflection is a MR, to 38.2° , 34.5° , 33.9° , 32.5° , 31.8° , and finally to 30° for $M_s = 1.47$ and $\gamma = \frac{5}{3}$ (these values are chosen for illustrative purposes only).

Figures 8(a)–8(g) represent the $(I-R)$ -polar combinations of the just-mentioned eight cases. Figure 8(a) shows the $(I-R)$ -polar combination for $\theta_w^C = 41^\circ$ ($\phi_1 = 49^\circ$). The intersection of the I and R polars yields a standard solution of the 3ST ($\theta_1 - \theta_2 = \theta_3$), and hence the resulting wave configuration is a MR. Figure 8(a) also reveals that the I and R polars intersect at their strong segments, i.e., along their subsonic branches. Hence, the flows on both sides of the slipstream of the MR near the triple point are subsonic ($M_2 < 1$ and $M_3 < 1$). This implies that in the case of a pseudosteady flow, the MR is a single MR.²⁹

When θ_w^C is reduced to $\theta_w^C = 38.2^\circ$ ($\phi_1 = 51.8^\circ$), the situation in which $\theta_2 = 0$ (i.e., $\theta_1 \pm \theta_2 = \theta_3$ or $\theta_1 = \theta_3$) is reached [see Fig. 8(b)]. As mentioned earlier, this is the point where the MR terminates and the vNR forms, i.e., this is the MR $\leftrightarrow$ vNR transition point.¹¹ Figure 8(b) also reveals that this point also yields $M_2 < 1$ and $M_3 < 1$. The wave configuration corresponding to this situation is presented in Fig. 6(c).

Figure 8(c) shows the $(I-R)$ -polar combination for $\theta_w^C = 34.5^\circ$ ($\phi_1 = 55.5^\circ$). The intersection of the I and R polars results in a nonstandard solution of the 3ST ($\theta_1 + \theta_2 = \theta_3$). Hence, the resulting reflection is a vNR. Figure 8(c) also reveals that the flows on both sides of the slipstream of the vNR are subsonic ($M_2 < 1$ and $M_3 < 1$) near the triple point. A numerically calculated wave configuration of a vNR is shown in Fig. 3(a).

When θ_w^C is reduced to $\theta_w^C = 33.9^\circ$ ($\phi_1 = 56.1^\circ$), the situation shown in Fig. 8(d) is reached. Now the R polar intersects the subsonic branch of the I polar exactly at its sonic point; hence, $M_2 = 1$ and $M_3 < 1$ near the triple point. This situation corresponds to the point beyond which the 3ST has a solution which is not physical! Hence, this is the point where the vNR terminates and gives rise to another wave configuration that was earlier termed ?R. A numerical calculation of this wave configuration is shown in Fig. 3(b). Thus, the situation in Fig. 8(d) represents the point at which the vNR terminates and the ?R forms, i.e., the vNR $\leftrightarrow$?R transition point, and also the point at which solutions of the 3ST become nonphysical, and the 3ST should be replaced by another theory, a 4WT that accounts for three shock waves and one expansion wave that are complemented by a slipstream. Hence, this situation also corresponds to the 3ST $\leftrightarrow$ 4WT transition.

When θ_w^C is further reduced to $\theta_w^C = 33.4^\circ$ ($\phi_1 = 56.6^\circ$), the situation shown in Fig. 8(e) is reached. The R polar still intersects the I polar, i.e., the 3ST still provides a solution. However, the polars intersect along their weak-solution branches, which means that the flow behind the reflected shock wave is supersonic, i.e., $M_2 > 1$. This solution implies that the reflected shock wave is directed toward the incoming flow and that the flow in state 2 is supersonic. This solution is not physical, since it implies that the wedge does not influence the part of the reflected shock wave above the triple point. The $(I-R)$ -polar combinations of the alternative 4WT, which results in the ?R [see Fig. 3(b)], is shown by the line

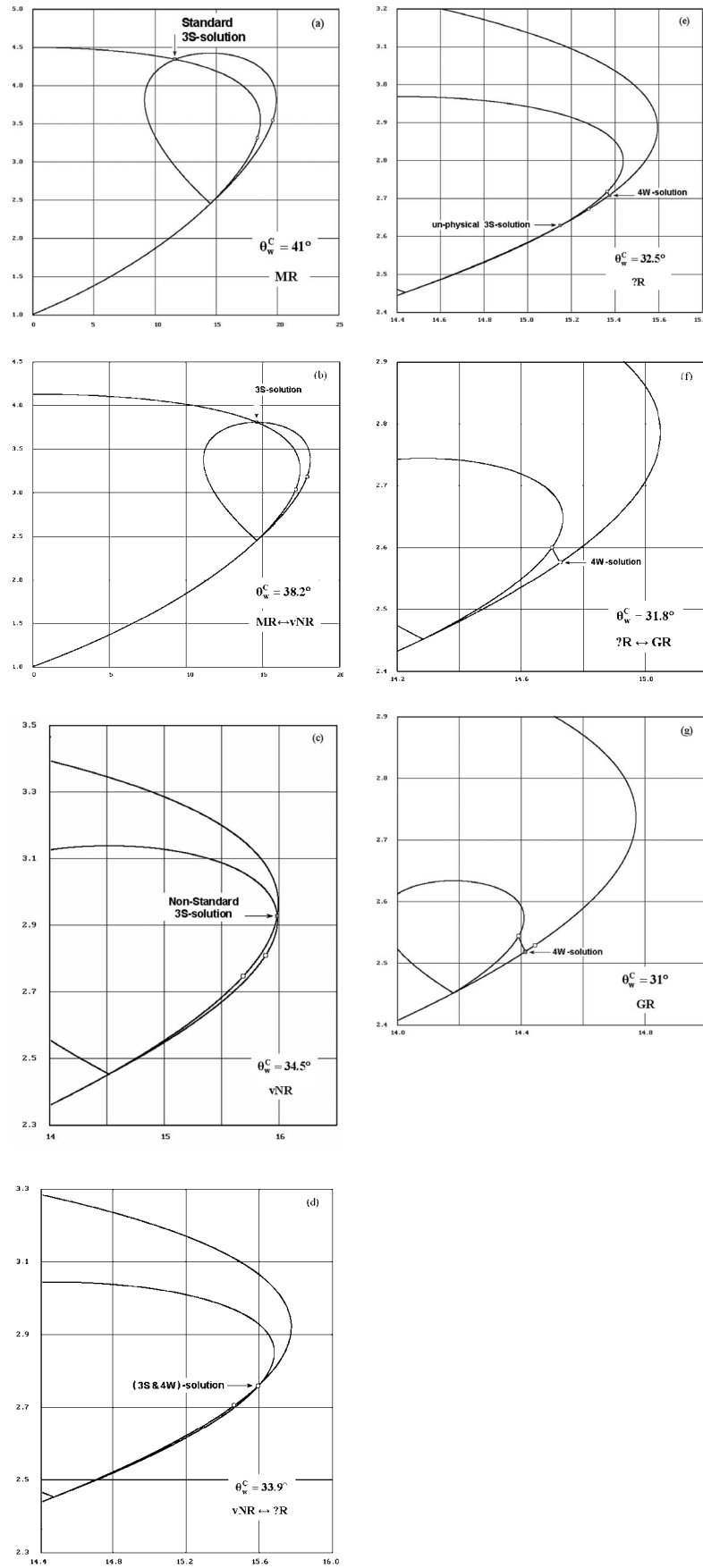


FIG. 8. The $(I-R)$ -polar combinations for $M_s = 1.47$ and $\gamma = \frac{5}{3}$: (a) $\theta_w^C = 41^\circ$ (MR); (b) 38.2° (MR $\leftrightarrow$ vNR); (c) 34.5° (vNR); (d) 33.9° (MR $\leftrightarrow$ vNR); (e) 32.5° (?R); (f) 31.8° (?R $\leftrightarrow$ GR); and (g) 31° (GR). Recall that $\theta_w^C = 90^\circ - \phi_1$. The sonic point is marked on all the shock polars.

TABLE I. Summary of the reflection types that occur when θ_w^C is decreased for $M_s=1.47$ and $\gamma=\frac{5}{3}$.

θ_w^C and ϕ_1	Mach number in state 2 near the triple point	Mach number in state 3 near the triple point	Reflection configuration and transition criteria
$\theta_w^C=41.0^\circ$ and $\phi_1=49.0^\circ$	$M_2 < 1$	$M_3 < 1$	MR
$\theta_w^C=38.2^\circ$ and $\phi_1=51.8^\circ$		$\phi_2=90^\circ$	MR $\leftrightarrow$ vNR
$\theta_w^C=34.5^\circ$ and $\phi_1=55.5^\circ$	$M_2 < 1$	$M_3 < 1$	vNR
$\theta_w^C=33.9^\circ$ and $\phi_1=56.1^\circ$		$M_2=1$	vNR $\leftrightarrow$?R also 3ST $\leftrightarrow$ 4WT
$\theta_w^C=33.4^\circ$ and $\phi_1=56.6^\circ$	$M_2=1$	$M_3 < 1$	?R
$\theta_w^C=31.8^\circ$ and $\phi_1=58.2^\circ$		$M_3=1$	?R $\leftrightarrow$ GR
$\theta_w^C=31.0^\circ$ and $\phi_1=59.0^\circ$	$M_2=1$	$M_3 > 1$	GR

bridging a subsonic state on the *I* polar (state 3) with the sonic point of the *R* polar (state 2). The fact that a Prandtl–Meyer expansion fan exists between two flow zones which are not supersonic can be explained as follows. A Prandtl–Meyer expansion fan cannot exist in a homogeneous subsonic flow. However, in the case under consideration, there is a strongly inhomogeneous converging flow ahead of the fan, and therefore the Prandtl–Meyer expansion fan can exist. If we consider two adjacent stream lines, the flow between them is similar to the flow inside the Laval nozzle with the minimum cross section at the boundary of the expansion fan.

When θ_w^C is reduced to $\theta_w^C=31.8^\circ$ ($\phi_1=58.2^\circ$), the situation shown in Fig. 8(f) is reached. Now the *I* and *R* polars do not intersect at all, and the 3ST does not provide any solution. Based on the 4WT,²³ which was presented earlier, the sonic points of the *I* and the *R* polars are bridged; hence, $M_2=1$ and $M_3=1$ near the triple point. In fact, this is the point where the ?R terminates and the modified GR forms, i.e., the ?R $\leftrightarrow$ GR transition point. Note that at this point, the *R* polar is tangent to the *I* polar at its origin.

It should be noted here that the Mach numbers in Table I, in the cases of ?R and GR, are near the triple point. As can be seen in Figs. 3(b) and 3(c), as the flows move away from the triple point, their Mach numbers change. In the cases of ?R and GR, the flow after the reflected shock wave changes from sonic to supersonic at transition, and in the case of a GR, the flow behind the Mach stem changes from supersonic to sonic. The sonic line is thus adjacent to the reflected shock and the flow does not need to accelerate to sonic conditions through a subsonic region. Thus, the transition boundaries determined should be accurate. For conditions away from the transition boundaries, however, account will need to be taken of the converging subsonic flow behind the reflected shock, as shown in Fig. 3.

A further reduction of θ_w^C to $\theta_w^C=31^\circ$ ($\phi_1=59^\circ$) results in the situation shown in Fig. 8(g). The *I* and *R* polars do not intersect, and the 3ST does not provide any solution. Based on the 4WT,²³ which was presented earlier, the *I* and the *R* polars are bridged at $M_2=1$ and $M_3>1$. The numerically obtained wave configuration of a modified GR is shown in

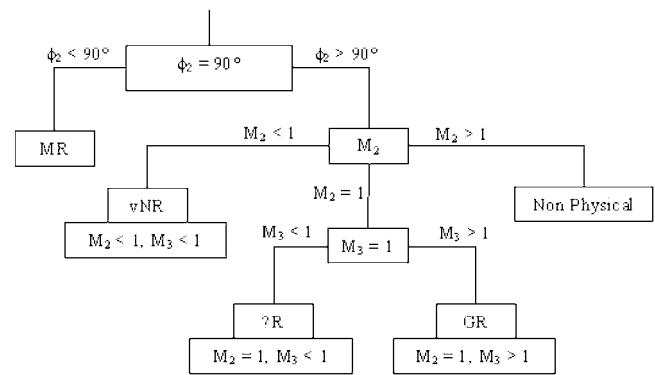


FIG. 9. Evolution-tree-type presentation of the transition criteria between the various reflections.

Fig. 3(c) and sketched schematically in Fig. 4. A Prandtl–Meyer expansion fan bridges the flows in states 2 and 3 whose conditions are sufficient to support a Prandtl–Meyer expansion fan.

The above-presented evolution of the types of reflection that are encountered when θ_w^C is reduced for $M_s=1.47$ and $\gamma=\frac{5}{3}$ and the transition criteria between them are summarized in Table I. It should be noted here that Henderson³⁰ had calculated some of these transition lines while mathematically investigating possible solutions of the 3ST. However, neither he nor anyone else attributed physical importance to these mathematical boundaries. In the present study, we suggest a full picture of the reflection phenomenon in the nonstandard-3ST domain and beyond it. The flow patterns described in the present study are consistent with all the available numerical and experimental data. The transition criteria of Table I are shown in Fig. 9 in an evolution-tree-type presentation.

Based on the foregoing discussion, Figs. 10(a) and 10(b) show the domains of and the transition boundaries between the various shock wave reflection configurations in the (M_s, θ_w^C) plane for (a) a diatomic ($\gamma=\frac{7}{5}$) and (b) a monatomic ($\gamma=\frac{5}{3}$) gas, respectively. Curve 1 is the MR $\leftrightarrow$ vNR transition curve, i.e., $\phi_2=90^\circ$ on this curve. Above this curve, $\phi_2 < 90^\circ$ and the reflection is a MR. Curve 2 is the vNR $\leftrightarrow$?R transition curve, i.e., $M_2=1$ on this curve. It should be noted here that this curve is different from the $M_2=1$ on the left branch of the *R* polar, which is the sonic criterion for the RR $\leftrightarrow$ IR transition. This curve also separates between the domains in which the 3ST does or does not have physical solutions, i.e., it corresponds to the 3ST $\leftrightarrow$ 4WT transition. Curve 3 is the ?R $\leftrightarrow$ GR transition curve, i.e., $M_3=1$ on this curve. Curve 4 is the curve $M_1=1$. Below this curve, the flow behind the incident shock wave is subsonic and no reflection can take place. The domain below this curve is sometimes referred to as the no-reflection domain (NR domain). The NR domain exists only in the (M_s, θ_w^C) plane. It vanishes in the more physical (M_s, θ_w) plane (recall that $\theta_w^C = \theta_w + \chi$ where χ is the triple-point trajectory angle). Curve 5 divides the (M_s, θ_w^C) plane into two domains: Above it, the 3ST has at least one solution (not necessarily physical) and below it, the

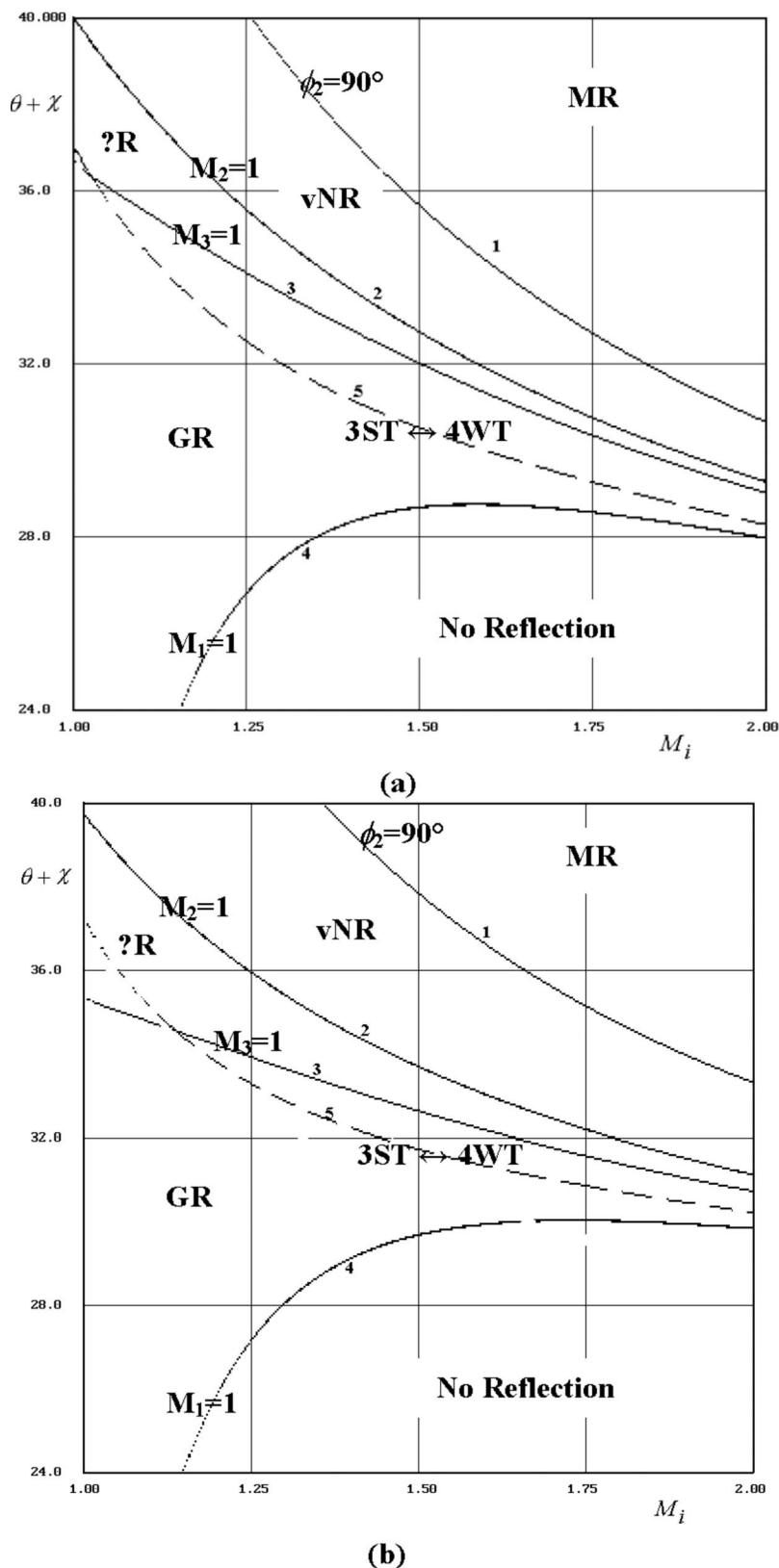


FIG. 10. Domains of and the transition boundaries between the various shock wave reflection configurations in the $(M_i, \theta_w + \chi)$ plane for (a) $\gamma = \frac{7}{5}$ and (b) $\gamma = \frac{5}{3}$. Curve 1: The MR $\leftrightarrow$ vNR transition curve, i.e., $\phi_2 = 90^\circ$. Curve 2: The vNR $\leftrightarrow$?R transition curve, i.e., $M_2 = 1$. This curve also separates the domains in which the 3ST does or does not have a physical solution. Curve 3: The ?R $\leftrightarrow$ GR transition curve, i.e., $M_3 = 1$. Curve 4: The curve on which $M_1 = 1$. Below this curve, the flow behind the incident shock wave is subsonic and a reflection cannot take place! Curve 5: Above this curve, the 3ST has at least one mathematical solution (not necessarily physical) and below it, the 3ST does not have any mathematical solution ($\theta_w^c = \theta_w + \chi = \pi/2 - \phi_1$, $M_i = M_0 \sin \phi_1$, M_i is the incident shock wave Mach number, θ_w is the reflecting wedge angle, and χ is the triple-point trajectory angle).

3ST does not have any solution. Consequently, between curves 2 and 5, the 3ST has a solution, which is not physical. The von Neumann paradox existed inside the domain bounded by curves 2 and 4. Guderley⁵ resolved the paradox in the domain bounded by curves 3 and 4 by forwarding the four-wave concept (three shock waves and one expansion

wave). The reflection in this domain is a GR [see Fig. 3(c)]. The reflection that occurs inside the domain bounded by curves 2 and 3 is ?R and is shown in Fig. 3(b). To the best of our knowledge, this wave configuration was not considered previously. In summary, the MR domain is above curve 1, the vNR domain is located between curves 1 and 2, the ?R

domain is located between curves 2 and 3, the modified GR domain is located between curves 3 and 4, and the NR domain is below curve 4. The 3ST has at least one solution (not necessarily physical) in the domain above curve 5 and no solutions in the domain below curve 5. Chapman³¹ who followed Henderson's³⁰ mathematical investigation drew curves 1, 2, and 4 (see Fig. 11.2.3a in his book) in the (M_0, ϕ_1) domain without giving them their physical importance and interpretation as has been done in the present study.

SUMMARY

The reflection of weak shock waves has been reconsidered analytically using shock polars and all possible flow patterns were analyzed.

It has been pointed out that the vNR concept that was forwarded by Colella and Henderson¹¹ as a resolution of the von Neumann paradox is not correct, since even according to them, the vNR occurs at conditions for which the 3ST does have a solution.

Based on the boundary condition across the slipstream of a three-shock confluence, the solutions of the 3ST were classified as "standard-3ST solutions" and "nonstandard-3ST solutions." When the 3ST has a standard solution, the reflection is a MR.

It was shown that there are two situations in the non-standard case:

- A situation where the 3ST has solutions of which at least one is physical. The wave configuration in this case is a vNR.
- A situation where the 3ST has a solution, which is not physical, and hence no reflection having a three-shock confluence is possible. A new wave configuration, ?R, occurs in this case.

In addition, it was shown that there are cases when the 3ST does not have any solution. The wave configuration in this case is a modified GR. It should be noted here that although we use the term Guderley reflection, the wave configuration that was originally suggested by Guderley^{5,6} did not include a slipstream (see Fig. 7), while the present numerical investigation revealed a wave configuration that includes a slipstream (see Fig. 4).

In the cases where the 3ST does not provide a physical or any solution, it is replaced by a 4WT, which was also presented in this study.

Both the ?R and the modified GR include a Prandtl-Meyer expansion fan in addition to the well known three shock waves of a MR. ?R and the GR differ in the number of their supersonic patches. Numerically obtained wave con-

figurations of vNR, a ?R, and a modified GR are shown in Figs. 3(a)–3(c), respectively. Note that while the GR was suggested by Guderley⁵ and modified in this study in order to resolve the von Neumann paradox, the ?R was first suggested by Vasilev.²³

Finally, the domains and transition boundaries between the four types of reflection that can take place in the weak shock domain, MR, vNR, ?R, and GR, were elucidated.

For the reader's convenience, formulas for calculating the transition lines between MR, vNR, ?R, and modified GR that are shown in Fig. 10 are presented in the Appendix.

ACKNOWLEDGMENTS

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APPENDIX: FORMULAS FOR THE TRANSITION BOUNDARIES

Using the expression for the Mach number, $M = u / \sqrt{\gamma RT}$, where $\gamma = C_p / C_v$ is the specific heat capacity ratio and R is the specific gas constant, the eight parameters of Eqs. (1a)–(1d) become $p_i, p_j, T_i, T_j, M_i, M_j, \phi_j$, and θ_j . Assuming that p_i, T_i, M_i , and ϕ_j are known results in the following solution of the conservation equations across an oblique shock wave:

$$\frac{p_j}{p_i} = 1 + \frac{2\gamma}{\gamma+1} (M_i^2 \sin^2 \phi_j - 1), \quad (\text{A1a})$$

$$\frac{p_j}{\rho_i} = \frac{\gamma+1}{(\gamma-1) + \frac{2}{M_i^2 \sin^2 \phi_j}}, \quad (\text{A1b})$$

$$M_j^2 = \frac{\left[1 + \frac{1}{2}(\gamma-1)M_i^2 \sin^2 \phi_j\right]^2 + \left[\frac{1}{4}(\gamma+1)M_i^2 \sin 2\phi_j\right]^2}{\left[1 + \frac{1}{2}(\gamma-1)M_i^2 \sin^2 \phi_j\right] \left[\gamma M_i^2 \sin^2 \phi_j - \frac{1}{2}(\gamma-1)\right]}, \quad (\text{A1c})$$

$$\theta_j = \tan^{-1} \frac{(M_i^2 \sin^2 \phi_j - 1) \cot \phi_j}{1 + M_i^2 \left[\frac{1}{2}(\gamma+1) - \sin^2 \phi_j\right]}. \quad (\text{A1d})$$

Shock polars

Equations (A1a) and (A1d) provide the (p, θ) polar for a supersonic flow with Mach number M_i in a parametric form. The full shock polar can be obtained by changing the parameter ϕ_j in the range $\sin^{-1}(1/M_i) \leq \phi_j \leq \pi - \sin^{-1}(1/M_i)$. The expression for the slope of the (p, θ) polar, $dp/d\theta = (dp/d\phi)/(d\theta/d\phi)$, reads

$$\frac{1}{p_i} \frac{dp_j}{d\theta_j} = \frac{\left[\frac{2\gamma}{(\gamma+1)}\right] \left\{ (\gamma M_i \sin \phi_j)^4 - \left[(\gamma-1) + \frac{1}{4}(\gamma+1)M_i^2 \right] (M_i \sin \phi_j)^2 - 1 \right\} M_i^2 \sin 2\phi_j}{\gamma (M_i \sin \phi_j)^4 + 2 \left[1 - \frac{1}{4}(\gamma+1)M_i^2 \right] (M_i \sin \phi_j)^2 - \left[1 + \frac{1}{2}(\gamma+1)M_i^2 \right]}. \quad (\text{A2})$$

Solution of the three-shock theory

For the incident shock wave ($j=1$ and $i=0$, see Fig. 2), Eqs. (A1a), (A1c), and (A1d) read

$$\frac{p_1}{p_0} = 1 + \frac{2\gamma}{\gamma+1}(M_0^2 \sin^2 \phi_1 - 1), \quad (\text{A3a})$$

$$\theta_1 = \tan^{-1} \frac{(M_0^2 \sin^2 \phi_1 - 1) \cot \phi_1}{1 + M_0^2 \left[\frac{1}{2}(\gamma+1) - \sin^2 \phi_1 \right]}, \quad (\text{A3b})$$

$$M_1^2 = \frac{\left[1 + \frac{1}{2}(\gamma-1)M_0^2 \sin^2 \phi_1 \right]^2 + \left[\frac{1}{4}(\gamma+1)M_0^2 \sin 2\phi_1 \right]^2}{\left[1 + \frac{1}{2}(\gamma-1)M_0^2 \sin^2 \phi_1 \right] \left[\gamma M_0^2 \sin^2 \phi_1 - \frac{1}{2}(\gamma-1) \right]}. \quad (\text{A3c})$$

For the reflected shock wave ($j=2$ and $i=1$), Eqs. (A1a), (A1c), and (A1d) yield

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 \sin^2 \phi_2 - 1), \quad (\text{A4a})$$

$$\theta_2 = \tan^{-1} \frac{(M_1^2 \sin^2 \phi_2 - 1) \cot \phi_2}{1 + M_1^2 \left[\frac{1}{2}(\gamma+1) - \sin^2 \phi_2 \right]}, \quad (\text{A4b})$$

$$M_2^2 = \frac{\left[1 + \frac{1}{2}(\gamma-1)M_1^2 \sin^2 \phi_2 \right]^2 + \left[\frac{1}{4}(\gamma+1)M_1^2 \sin 2\phi_2 \right]^2}{\left[1 + \frac{1}{2}(\gamma-1)M_1^2 \sin^2 \phi_2 \right] \left[\gamma M_1^2 \sin^2 \phi_2 - \frac{1}{2}(\gamma-1) \right]}. \quad (\text{A4c})$$

Similarly, for the Mach stem ($j=3$ and $i=0$), one obtains

$$\frac{p_3}{p_0} = 1 + \frac{2\gamma}{\gamma+1}(M_0^2 \sin^2 \phi_3 - 1), \quad (\text{A5a})$$

$$\theta_3 = \tan^{-1} \frac{(M_0^2 \sin^2 \phi_3 - 1) \cot \phi_3}{1 + M_0^2 \left[\frac{1}{2}(\gamma+1) - \sin^2 \phi_3 \right]}, \quad (\text{A5b})$$

$$M_3^2 = \frac{\left[1 + \frac{1}{2}(\gamma-1)M_0^2 \sin^2 \phi_3 \right]^2 + \left[\frac{1}{4}(\gamma+1)M_0^2 \sin 2\phi_3 \right]^2}{\left[1 + \frac{1}{2}(\gamma-1)M_0^2 \sin^2 \phi_3 \right] \left[\gamma M_0^2 \sin^2 \phi_3 - \frac{1}{2}(\gamma-1) \right]}. \quad (\text{A5c})$$

The pressure boundary condition across the slipstream ($p_2 = p_3$) can be rewritten as follows:

$$\frac{p_3}{p_0} = \frac{p_2}{p_1} \frac{p_1}{p_0}. \quad (\text{A6})$$

Inserting Eqs. (A3a), (A4a), and (A5a) into Eq. (A6) yields the following formula:

$$\sin^2 \phi_3 = \frac{(\gamma^2 - 1) + [2\gamma M_0^2 \sin^2 \phi_1 - (\gamma - 1)][2\gamma M_1^2 \sin^2 \phi_2 - (\gamma - 1)]}{2\gamma(\gamma + 1)M_0^2}. \quad (\text{A7})$$

Inserting relations (A3b), (A4b), (A5b), (A3c), (A4c), and (A7) into the second boundary condition at the slipstream, $\theta_1 \pm \theta_2 = \theta_3$ [Eq. (3)] results in a single implicit equation for ϕ_2 as a function of M_0 and ϕ_1 .

Equations for calculating the transition lines in Fig. 10

In the following, closed sets of equations, which are all based on the above equations, for calculating the various transition lines in Fig. 10 are given.

Line 1: The MR ↔ vNR transition

On this line, $\phi_2 = \pi/2$, i.e., $\sin \phi_2 = 1$ and $\theta_2 = 0$. Hence, the flow-deflection boundary condition reduces to

$$\theta_1 = \theta_3 \quad (\text{A8})$$

Inserting $\sin \phi_2 = 1$ into Eq. (A7) yields

$$\sin^2 \phi_3 = \frac{(\gamma^2 - 1) + [2\gamma M_0^2 \sin^2 \phi_1 - (\gamma - 1)][2\gamma M_1^2 - (\gamma - 1)]}{2\gamma(\gamma + 1)M_0^2}. \quad (\text{A9})$$

Inserting $M_1 = M_1(M_0, \phi_1)$ from Eq. (A3c) into Eq. (A9) yields

$$\phi_3 = \phi_3(M_0, \phi_1).$$

Inserting this relation into Eq. (A5b) yields the following relation:

$$\theta_3 = \theta_3(M_0, \phi_1). \quad (\text{A10})$$

Inserting the latter relation into Eq. (A8) together with Eq. (A3b), one obtains the formula for the MR $\leftrightarrow$ vNR transition line (line 1 in Fig. 10):

$$\theta_3(M_0, \phi_1) = \theta_2(M_0, \phi_1). \quad (\text{A11})$$

Line 2: The vNR $\leftrightarrow$?R transition

On this line, $M_2=1$. Inserting $M_1(M_0, \phi_1)$ from Eq. (A3c) into Eq. (A4c) and applying the condition $M_2=1$ results in the following relation:

$$\phi_2 = \phi_2(M_0, \phi_1).$$

Inserting the latter relation together with $M_1(M_0, \phi_1)$ from Eq. (A3c) into Eqs. (A4b) and (A7) yields the following formulas:

$$\begin{aligned} \phi_3 &= \phi_3(M_0, \phi_1), \\ \theta_2 &= \theta_2(M_0, \phi_1), \end{aligned} \quad (\text{A12})$$

Inserting $\phi_3 = \phi_3(M_0, \phi_1)$ into Eq. (A5b) yields the following relation:

$$\theta_3 = \theta_3(M_0, \phi_1). \quad (\text{A13})$$

Inserting Eqs. (A3b), (A12), and (A13) into the flow-deflection boundary condition of a nonstandard solution re-

sults in the formula for the vNR $\leftrightarrow$?R transition line (line 2 in Fig. 10):

$$\theta_3(M_0, \phi_1) = \theta_1(M_0, \phi_1) + \theta_2(M_0, \phi_1). \quad (\text{A14})$$

Line 3: The ?R $\leftrightarrow$ GR transition

Equations (1) and (5)–(8) imply the dependence $M_3 = M_3(M_0, \phi_1)$. The equation of line 3 for the ?R $\leftrightarrow$ GR transition is $M_3=1$.

Line 5: The 3ST $\leftrightarrow$ 4WT transition

In this case, the R polar is tangent to the I polar at the transition. Hence, the slopes of the R and the I polars are identical at the point where the R polar emanates from the I polar. This condition implies that

$$\left. \frac{dp_I}{d\theta} \right|_1 = \left. \frac{dp_R}{d\theta} \right|_1,$$

where the subscripts I and R refer to the I and R polars, respectively, and 1 indicates the point on the I polar from which the R polar emanates. The slope of a general (p, θ) polar is given by Eq. (A2). Applying this equation to the I polar results in the following formula:

$$\left. \frac{1}{p_0} \frac{dp_I}{d\theta} \right|_1 = \frac{\left[\frac{2\gamma}{(\gamma+1)} \right] \left\{ \gamma(M_0 \sin \phi_1)^4 - \left[(\gamma-1) + \frac{1}{4}(\gamma+1)^2 M_0^2 \right] (M_0 \sin \phi_1)^2 - 1 \right\} M_0^2 \sin 2\phi_1}{\gamma(M_0 \sin \phi_1)^4 + 2 \left[1 - \frac{1}{4}(\gamma+1) M_0^2 \right] (M_0 \sin \phi_1)^2 - \left[1 + \frac{1}{2}(\gamma+1) M_0^2 \right]}. \quad (\text{A15})$$

Similarly, applying Eq. (A2) to the R polar yields

$$\left. \frac{1}{p_1} \frac{dp_R}{d\theta} \right|_1 = \frac{\left[\frac{2\gamma}{(\gamma+1)} \right] \left\{ \gamma(M_1 \sin \phi_2)^4 - \left[(\gamma-1) + \frac{1}{4}(\gamma+1)^2 M_1^2 \right] (M_1 \sin \phi_2)^2 - 1 \right\} M_1^2 \sin 2\phi_2}{\gamma(M_1 \sin \phi_2)^4 + 2 \left[1 - \frac{1}{4}(\gamma+1) M_1^2 \right] (M_1 \sin \phi_2)^2 - \left[1 + \frac{1}{2}(\gamma+1) M_1^2 \right]}. \quad (\text{A16a})$$

Multiplying Eq. (A16a) by Eq. (A3a), one obtains

$$\begin{aligned} \left. \frac{1}{p_0} \frac{dp_R}{d\theta} \right|_1 &= \left[1 + \frac{2\gamma}{\gamma+1} (M_0^2 \sin^2 \phi_1 - 1) \right] \\ &\times \frac{\left[\frac{2\gamma}{(\gamma+1)} \right] \left\{ \gamma(M_1 \sin \phi_2)^4 - \left[(\gamma-1) + \frac{1}{4}(\gamma+1)^2 M_1^2 \right] (M_1 \sin \phi_2)^2 - 1 \right\} M_1^2 \sin 2\phi_2}{\gamma(M_1 \sin \phi_2)^4 + 2 \left[1 - \frac{1}{4}(\gamma+1) M_1^2 \right] (M_1 \sin \phi_2)^2 - \left[1 + \frac{1}{2}(\gamma+1) M_1^2 \right]}. \end{aligned} \quad (\text{A16b})$$

The slope on the R polar is measured at its origin; hence, at that point, $M_1 \sin \phi_2=1$. Inserting this value into Eq. (A16b) yields

$$\left. \frac{1}{p_0} \frac{dp_R}{d\theta} \right|_1 = \left[1 + \frac{2\gamma}{\gamma+1} (M_0^2 \sin^2 \phi_1 - 1) \right] \frac{\left[\frac{2\gamma}{(\gamma+1)} \right] \left\{ \gamma - \left[(\gamma-1) + \frac{1}{4}(\gamma+1)^2 M_1^2 \right] - 1 \right\} 2(M_1^2 - 1)^{1/2}}{\gamma + 2 \left[1 - \frac{1}{4}(\gamma+1) M_1^2 \right] - \left[1 + \frac{1}{2}(\gamma+1) M_1^2 \right]}. \quad (\text{A16c})$$

Using Eq. (A3c), the right hand side of Eq. (A16c) can be expressed as a function of M_0 and ϕ_1 only. Equating the slopes given by Eqs. (A15) and (A16c) results in the formula for the 3ST $\leftrightarrow$ 4WT transition line (line 5 in Fig. 10).

Line 4: The GR↔no-reflection transition

The relation for line 4 below for which the flow behind the incident shock wave is subsonic, and hence no reflection can take place, is as follows:³¹

$$\sin^2 \phi_1 = \frac{\frac{1}{2}(\gamma - 3) + \frac{1}{2}(\gamma + 1)M_0^2 + (\gamma + 1)^{1/2} \left[\frac{1}{4}(\gamma + 9) + \frac{1}{2}(\gamma - 3)M_0^2 + \frac{1}{4}(\gamma + 1)M_0^4 \right]^{1/2}}{2\gamma M_0^2}. \quad (\text{A17})$$

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