

ABOUT THE CHOICE OF UNIFORMLY DISTRIBUTED SEQUENCES TO BE USED IN THE RANDOM CHOICE METHOD

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The numerical aspects of the random choice method for the solution of the initial value problems of gasdynamics are discussed. The effect of different sampling schemes employed in the random choice method on the performance of the method is studied. A mathematical explanation for the obtained numerical results is offered.

1. Introduction

The random choice method (RCM), which has been used extensively in gasdynamics [1], was first employed by Glimm [2] as part of a constructive proof for the existence of weak solutions of systems of nonlinear hyperbolic equations.

The RCM gained ground over other numerical schemes, because it allows high resolution of shock waves and contact surfaces while in other finite difference methods these are usually smeared out over many grid points as a consequence of artificial viscosity and truncation error of the scheme. The RCM is based on Godunov's [3] suggestion to utilize the exact solution of the Riemann problem with piecewise constant initial data for the finite difference solution of hyperbolic equations in gasdynamics. In Godunov's method the solution $u(x_i, t_{n+1})$ at time t_{n+1} is obtained by solving a series of Riemann problems between neighboring points on a spatial grid with initial data obtained from the finite difference solution at time t_n . These solutions are then averaged over the intervals of the spatial grid in order to obtain the finite difference solution

$$u_i^{n+1} = \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} u^n(x, t_{n+1}) dx. \quad (1)$$

The value of time step $\Delta t_n = t_{n+1} - t_n$ is chosen to satisfy the Courant–Friedrichs–Lewy condition.

Godunov's original suggestion was modified by Glimm [2] and Chorin [4] who used the single-sample Monte Carlo estimate of the integral appearing in the right-hand side of (1), i.e.,

$$u_i^{n+1} = u^n(x^*, t_{n+1}), \quad (2)$$

where x^* is a point, sampled from the uniform probability density function $f(x)$. $f(x)$ is defined in the interval $[x_i, x_{i+1}]$ of length $h = x_{i+1} - x_i$ as follows:

$$f(x) = 1/(x_{i+1} - x_i). \quad (3)$$

This calculation of the finite difference solution of the gasdynamic equations at time $(n + 1)\Delta t$ (where Δt is a time step) from the finite difference solution at time $n \Delta t$ using Glimm's method can be viewed as a piecewise constant representation of the Monte Carlo estimate of the integral from the solution of the Riemann problem $u(x, t_{n+1})$ at time t_{n+1} .

The Riemann problem is solved repeatedly between each pair of neighboring grid points. The initial positioning of the discontinuities (shock waves, etc.) between these grid points is sampled with the help of random numbers, having uniform distribution in the interval $[-0.5, 0.5]$ (for details see [1, 4]). Various algorithms for generating these random numbers can be used. Apparently, the specific choice of a random number generator has a meaningful effect on the obtained numerical results, i.e., on the obtained numerical noise level. The purpose of the present paper is to demonstrate the relation between the adopted sampling procedure and the obtained numerical results. In addition, a plausible explanation for this relation is offered.

For demonstration purposes the case of a rarefaction wave interacting with a gradual area change in a duct was solved (this case was originally studied by Igra and Gottlieb [5]). The solution was repeated with four different sampling algorithms. The names given to these algorithms and a brief presentation of the employed sampling schemes are presented in the following:

- (1) IMSL sampling scheme;
- (2) Chorin's sampling scheme;
- (3) Lax's sampling scheme;
- (4) Van der Corput's sampling scheme.

2. The IMSL sampling scheme

The International Mathematical and Statistical Libraries (IMSL) sampling scheme employs the multiplicative congruential method [6] for generation of pseudorandom sequences α_i , uniformly distributed in the interval $[0, 1]$. The pseudorandom sequence is generated by the recurrence relation

$$\alpha_{i+1} = g\alpha_i \pmod{M}, \quad (4)$$

where $M = 2^{48}$. The multiplier g is an integer between 0 and $M - 1$. The values g and α_0 are chosen such that random numbers obtained by the recurrence relation (4) pass the Coveyou-Macpherson test as well as other statistical tests for randomness, including the autocorrelation test and the pair and triplet tests [6]. However, pseudorandom numbers obtained with generator (4) are known to be non equidistributed as required or, more correctly, desired for use in the RCM, in the sense discussed both by Chorin [4] and Colella [7]. Hence, (4) is not a good choice of a random number generator for the RCM. Actually, the implementation of the

IMSL sampling scheme in the RCM causes fast accumulation of numerical errors and explosion of the algorithm.

3. Chorin's sampling scheme

The goal of Chorin's sampling scheme is to produce a sequence of numbers with the aid of the IMSL sampling procedure,

$$\theta_i^{\text{IMSL}} = 2\alpha_i - 1, \quad (5)$$

which are equidistributed in the interval $[-0.5, 0.5]$. In order to improve the uniformity of quasirandom IMSL sequence (5), Chorin [4] has implemented the stratified sampling procedure [8]. The sampling scheme devised by Chorin is presented in the following. Let k_1 and k_2 be two mutually prime integers with $k_1 > k_2$. For example, let $k_1 = 11$ and $k_2 = 7$. Now consider the sequence of integers that is given by the following recurrence relation:

$$n_{i+1} = (n_i + k_2) \bmod(k_1), \quad (6)$$

for which the initial number n_0 is specified. For example, let us take $n_0 = 2$ and note that it must be less than k_2 . Based on this sequence of integers, a modified sequence having fairly well equidistributed numbers θ_i in the interval $[-0.5, 0.5]$ is given by the following expression:

$$\theta_i^{\text{Chorin}} = [(n_i + \alpha_i)/k_2] - 0.5, \quad (7)$$

where α_i is a sequence of pseudorandom numbers uniformly distributed in the interval $[0, 1]$ obtained with the IMSL sampling scheme (4).

4. Lax's sampling scheme

Lax proposed a sampling scheme which employs the uniformly nonrandom distributed sequences of points in the interval $[0, 1]$. The original suggestion of Lax was to use the following uniformly distributed sequence [7]:

$$\theta_i^{\text{Lax}} = (ir) \bmod(1) - 0.5, \quad (8)$$

where r is an irrational square root, i.e., $r = R^{1/2}$ with R an integer which is not the square of another integer. The uniformly distributed sequences of Lax (9) are a special case of the uniformly distributed sequences introduced by Weil [9] and studied extensively by others (the complete list of references is presented in [9]). Actually, Weil had proved that any sequence of numbers which can be expressed by the following recurrence relation,

$$x_n = (\alpha + nr) \bmod(1), \quad (9)$$

where α is any real number and r is an irrational number, is uniformly distributed in the interval $[0, 1]$. Therefore, the sequence of numbers

$$\theta_i^{\text{Weil}} = (\alpha + ir) \bmod(1) - 0.5 \quad (10)$$

is uniformly distributed in the interval $[-0.5, 0.5]$ and should be more suitable for use with the RCM than the original Chorin's sampling procedure with stratified sampling (7).

5. Van der Corput's sampling scheme

The uniformly distributed sequences of Van der Corput were first used with the RCM by Colella [7]. The choice was most fortunate, since among all known uniformly distributed sequences the deviation from uniformity is minimal in the Van der Corput and Sobol sequences [8, 9]. Another advantage of these sequences is that they can be extended to two or more dimensions. The complete presentation of the mathematical principle underlying the uniformly distributed sequences can be found in [9]. The sampling scheme of Van der Corput is described in the following. Suppose that R is any integer, then any other integer i can be written in radix- R notation as

$$i = \sum_{l=0}^L i_l R^l, \quad (11)$$

where $L = [\log_R i] = [\ln i / \ln R]$ and $[\cdot]$ denotes the integral part. By reversing the order of digits in i we can uniquely construct a fraction lying in the interval $[0, 1]$:

$$\alpha_i^R = \sum_{l=0}^L i_l R^{-l-1}. \quad (12)$$

Van der Corput's sequence of numbers, uniformly distributed in the interval $[0, 1]$, is obtained from expressions (11)–(12) with $R = 2$. Then the sequence of numbers

$$\theta_i^{\text{Van der Corput}} = 2\alpha_i^{(2)} - 0.5, \quad (13)$$

where $\alpha_i^{(2)}$ is obtained from (11)–(12) with $R = 2$, is distributed uniformly in the interval $[-0.5, 0.5]$ and should be better for use with the RCM than Chorin's or Lax's sampling schemes.

6. Numerical results and discussion

The numerical results obtained for the interaction of a rarefaction wave with an area enlargement in a duct using the above-mentioned sampling schemes are shown in Figs. 1–4. The results are presented in the form of separate sets of spatial distributions of dimensionless pressure p/p_1 and flow velocity u/a at successive time levels. Each successive distribution is

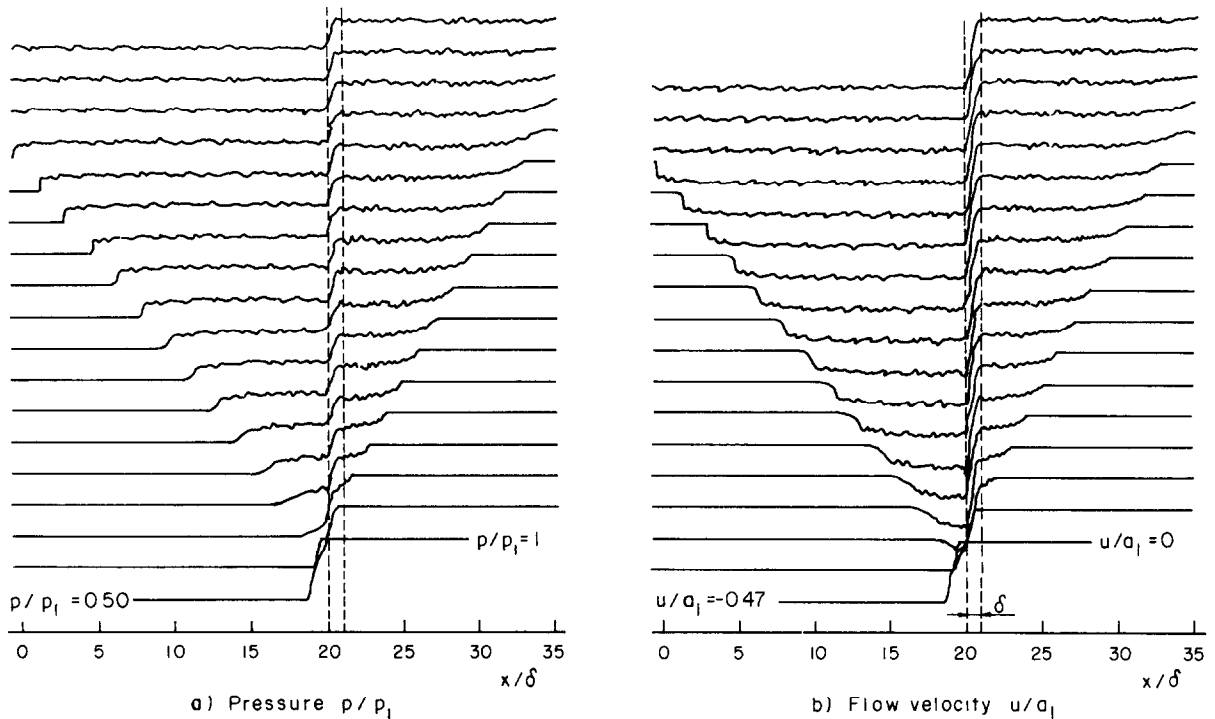


Fig. 1. Spatial distributions of (a) pressure and (b) flow velocity for the interaction of a rarefaction wave with an area enlargement (Chorin's sampling scheme).

placed slightly upward from the previous one in order to produce the effect of a time-distance diagram. The location of the area enlargement of length δ is marked by the two vertical dashed lines. The incident rarefaction wave is specified in the bottom distribution of each set, just to the left of the area enlargement. The flow field was computed with 720 grid points, of which 20 are specifically allocated to the area enlargement and 18 to the incident rarefaction wave profile. For the considered case the rarefaction wave strength was $p/p_1 = 0.50$ and the area enlargement ratio was $S_a/S_u = 0.20$.

The results obtained with Chorin's sampling scheme are presented in Fig. 1. As can be seen from this figure, a persistent numerical noise appears in the constant flow regions behind the transmitted rarefaction wave and between the upstream-facing downstream-swept reflected shock wave and the flow exit from the area enlargement. Employing the IMSL sampling scheme resulted in an intolerable numerical noise as can be seen from Fig. 2. A significant improvement over the first two results was obtained when Lax's sampling scheme was employed, as demonstrated in Fig. 3. As could be expected, the best results with minimum numerical noise were obtained with the use of Van der Corput's sampling scheme (Fig. 4).

In the following, a mathematical explanation of the numerical results is given. When the mesh used for the numerical solution becomes extremely fine, i.e., when the number of the mesh points $N \rightarrow \infty$ with $h/\Delta t = \text{const.}$, the Monte Carlo estimate (2) of the integral appearing in (1) converges to the u_i^n with probability one, since the number of points sampled in each mesh interval becomes infinitely large. The probabilistic error of such an estimate for finite N is of the order $O(N^{-1/2})$ [8, 9] which considerably exceeds the error of the finite difference

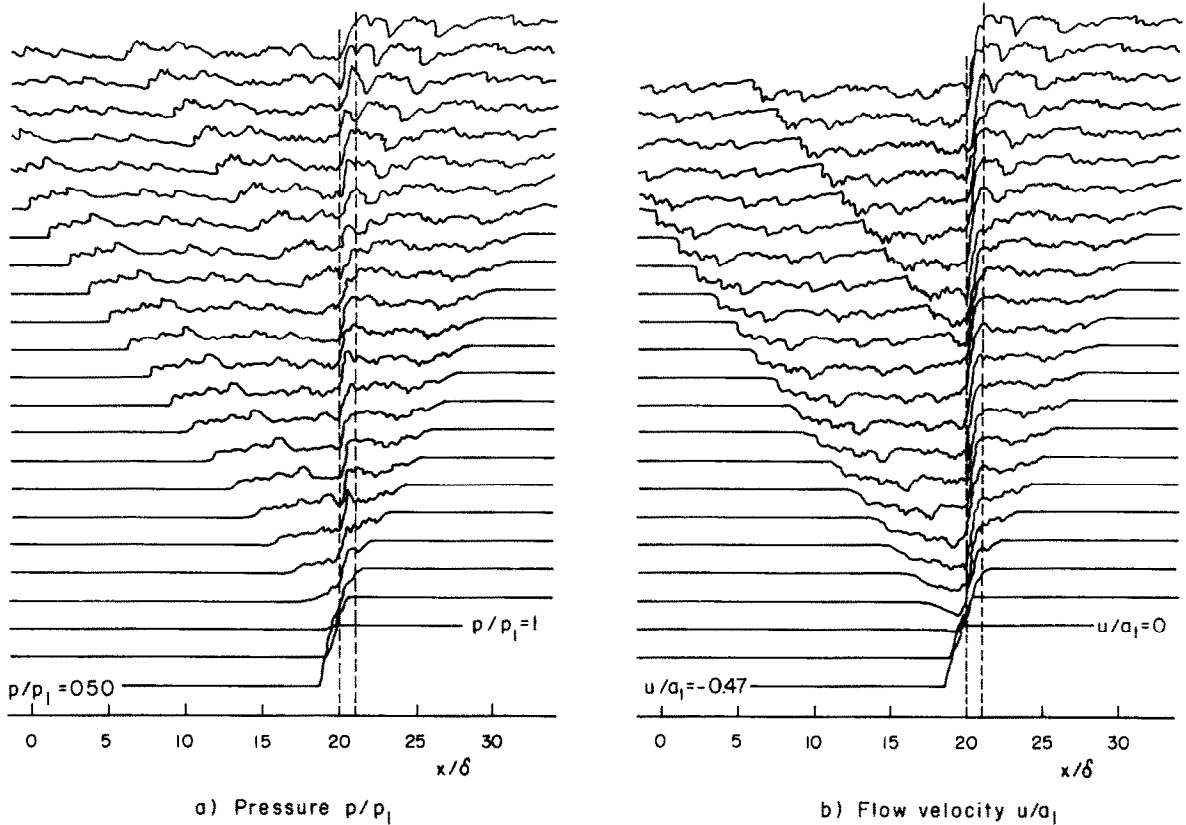


Fig. 2. Spatial distributions of (a) pressure and (b) flow velocity for the interaction of a rarefaction wave with an area enlargement (IMSL sampling scheme).

approximation, which is of order $O(1/N)$ [3]. However, the convergence of Glimm's construction is valid for any uniformly distributed sequence in the intervals $[x_i, x_{i+1}]$, as has been shown by Liu [10]. Thus, the value x^* in (2) has not necessarily to be sampled from a uniform probability density function (3), but can be chosen on the basis of purely deterministic considerations. The latter choice allows us to improve the convergence of the RCM, as has been demonstrated above.

The explanation of the fact that the application of uniformly distributed sequences in the RCM results in a considerable reduction of the numerical noise is as follows. When estimating the integral appearing in (1) by using uniformly distributed sequences, an accuracy of order $O((\log N)/N)$ is obtained. This error is asymptotically close to the error introduced by the finite difference scheme. The application of the uniformly distributed sequences (like Van der Corput's and/or Lax's sequences) in numerical integration is based on the theorem of Weil [9]. Weil was the first to define uniformly distributed sequences, to study their properties, and to find several constructive examples of such sequences in 1916. Weil's theorem states: If $\{P_i\}$ is a uniformly distributed sequence of points in a multidimensional polygon K^n , then for any function $f(P)$ integrable in the Riemann's sense, i.e., for any bounded function in an n -dimensional space, the following limiting relation is valid:

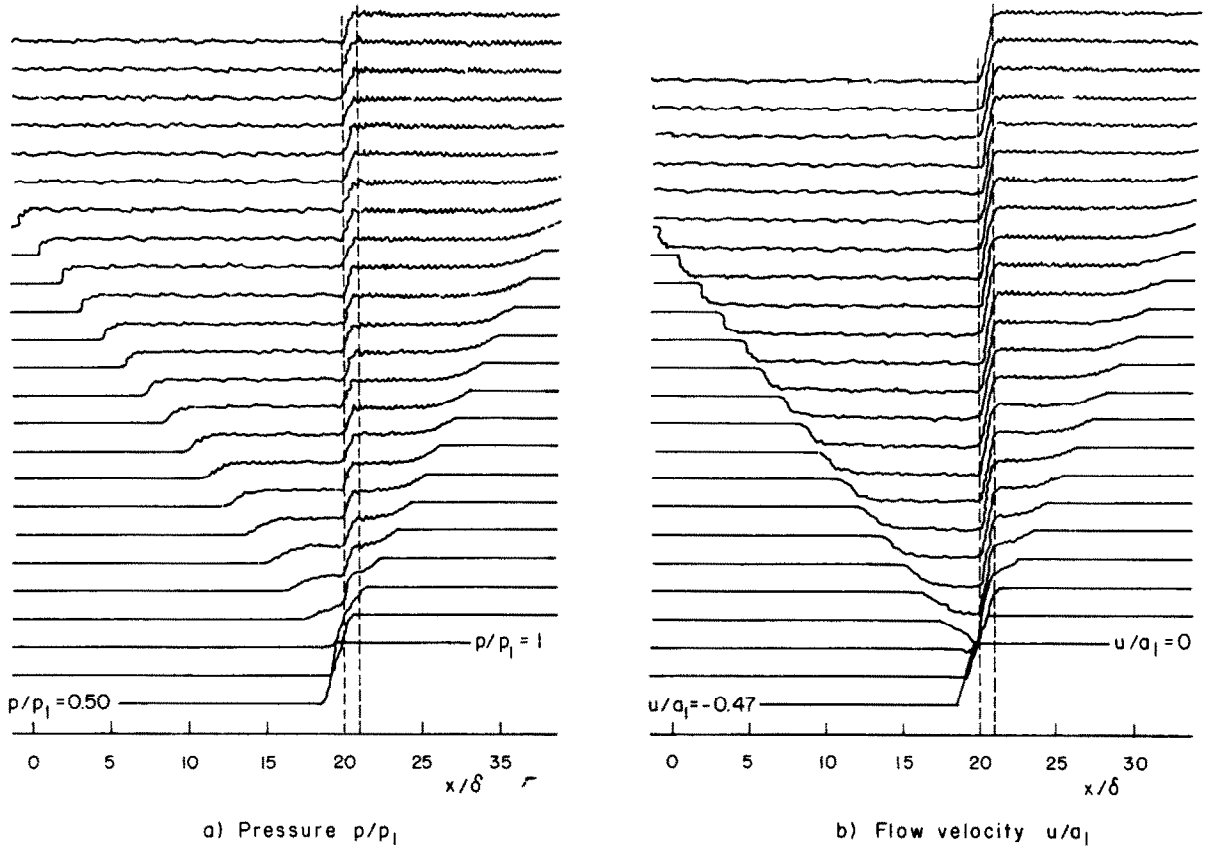


Fig. 3. Spatial distribution of (a) pressure and (b) flow velocity for the interaction of a rarefaction wave with an area enlargement (Lax's sampling scheme).

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N f(P_i) = \int_{K^n} f(P) dP. \quad (14)$$

If (14) is valid for any bounded function $f(P)$, then the sequence of points $\{P_i\}$ is uniformly distributed in K^n . It has been proven [9] that the rate of convergence in (14) is of order $O((\log N)/N)$. Therefore, the application of the uniformly distributed sequences allows us to obtain asymptotically the best possible rate of convergence, which cannot exceed $O(1/N)$ even for analytical functions according to the theorem of Centsov [9]. Thus, it can be concluded that the application of the uniformly distributed sequences of Van der Corput with the RCM resulted in the best possible accuracy, of order $O((\log N)/N)$, which is asymptotically close to the accuracy of any finite difference scheme of first order. This conclusion has important implications in the implementation of Glimm's method for solving gasdynamic problems.

The stochastic version of Glimm's method, which employs quasirandom sequences, has to be performed with a fine grid (and small time steps) in order to reduce the random error of order $O(1/\sqrt{N})$ to the level consistent with the error associated with the finite difference

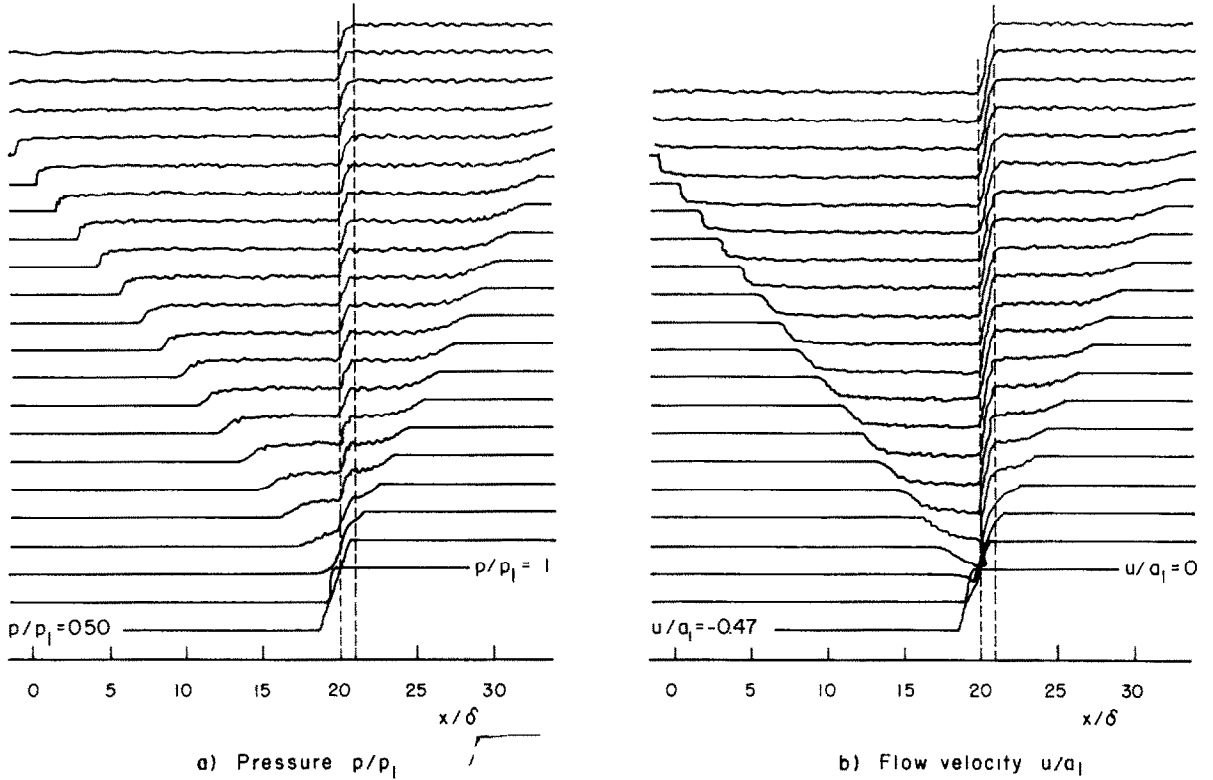


Fig. 4. Spatial distribution of (a) pressure and (b) flow velocity for the interaction of a rarefaction wave with an area enlargement (Van der Corput's sampling scheme).

approximation. Since the minimum error attained due to the implementation of uniformly distributed sequences in the deterministic version of Glimm's method is of order $O((\log N)/N)$, the value of the time step Δt_{RC} to be employed in the calculations has to be smaller than that resulting from the Courant–Friedrichs–Lewy condition Δt_{CFL} :

$$\Delta t_{CFL}/\Delta t_{RC} = O(\log N). \quad (15)$$

Thus, both the deterministic and the random versions of Glimm's method have to be implemented with smaller values of time steps than those imposed by the Courant–Friedrichs–Lewy condition.

7. Conclusions

An analysis of stochastic and deterministic versions of Glimm's method for the solution of initial value problems in gasdynamics has been performed. Four sampling procedures for the generation of uniformly distributed sequences have been analyzed in order to examine the quality of the solution. The uniformly distributed sequence of Van der Corput was found to provide the best results with a minimum level of numerical noise. The explanation of

numerical experiments based on the theory of uniformly distributed sequences has been suggested. The important conclusion which should be drawn from the presented analysis is that it is necessary to use smaller values of time steps than those resulting from the Courant–Friedrichs–Lewy condition in the applications of deterministic and random versions of Glimm’s method.

References

- [1] T. Saito and I.I. Glass, Application of random choice method to problems in gasdynamics, *Prog. Aerospace Sci.* 21 (1984) 201–247.
- [2] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* 18 (1965) 697–715.
- [3] S.K. Godunov, Finite difference method for numerical calculations of discontinuous solutions of equations of hydrodynamics, *Mat. Sb.* 47 (1959) 271–306 [in Russian].
- [4] J. Chorin, Random choice solution of hyperbolic systems, *J. Comput. Phys.* 22 (1976) 517–533.
- [5] O. Igra, J.J. Gottlieb and T. Saito, An analytical and numerical study of the interaction of rarefaction waves with area changes in ducts. Part 2: area enlargements, Rept. No. 282, Institute for Aerospace Studies, University of Toronto, 1984.
- [6] D.E. Knuth, *The Art of Computer Programming, Vol. 2: Seminumerical Algorithms* (Addison-Wesley, Reading, MA, 1969).
- [7] P. Colella, Glimm’s method for gas dynamics, *SIAM J. Sci. Statist. Comput.* 3 (1982) 76–110.
- [8] J.M. Hammersley and D.C. Handscomb, *Monte Carlo Methods* (Methuen, London, 1964).
- [9] I.M. Sobol, *Multidimensional Quadratures and Haar Functions* (Nauka, Moscow, 1969) [in Russian].
- [10] T.P. Liu, The deterministic version of the Glimm scheme, *Comm. Math. Phys.* 57 (1977) 135–148.