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Nondissipative shock waves in two-phase flows

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Abstract

It is shown that the Korteweg–de Vries equation which describes dissipationless processes can occur also in the system of equations of multiphase hydrodynamics when dissipation is compensated by external supply of energy. Therefore all the phenomena which are characteristic for the Korteweg–de Vries equation (solitons, periodic waves, nondissipative shock waves) can occur also in multiphase hydrodynamics. The study analyzes in particular the phenomenon of formation of nondissipative shock waves. It is shown that multiphase filtration is accompanied by formation of a continuously expanding region with small scale undamping oscillations of phase composition and velocity. The analysis uses the Korteweg–de Vries equation which is derived from the system of conservation laws describing multiphase hydrodynamics in porous media. Obtained results are relevant for the analysis of multiphase filtration (viscous fingering in the hydrocarbon recovery process) and in the hydrodynamics of fluidized bed (formation of bubbles).

1. Introduction

The flows of fluids through porous media are widespread in nature and encountered in various engineering applications. The typical examples include filtration of water and oil through the ground, reactant flows in catalytic packed bed reactors, flows in fluidized bed reactors etc. In this investigation we consider only some general properties of these flows. Obviously, the here analyzed processes of pattern formation in multiphase flows do not exhaust all the variety of the phenomena occurring in such flows.

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It is generally believed that filtration through porous media can be adequately described by the empirical Darcy law [1]:

$$\nabla P = -\frac{\mu}{k} V , \tag{1}$$

where k is a permeability of porous medium, μ is the dynamic viscosity of filtrating fluid, ∇P is the pressure gradient and V is the flow rate velocity. In the case of multiphase filtration the consequence of this equation is the formation of self-steepening kinematic shock waves [2]. Such behavior arises due to nonlinear dependence of the interphase friction upon the phase composition. Another important field of application of

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filtration theory is provided by geophysics. In geophysics filtration phenomena are of particular significance in the formation of sedimental rocks and also in magma dynamics [3]. The latter phenomenon can be viewed as magma filtration through the Earth's mantle which can be described as a very viscous incompressible matrix.

Eq. (1) can be interpreted as a stationary linear approximation of the Navier–Stokes equation with local friction:

$$\rho \frac{\partial V}{\partial t} = -\frac{\mu}{k} V - \nabla P . \tag{2}$$

If pressure changes sufficiently slowly so that

$$\left| \frac{k\rho}{\mu} \frac{1}{P} \frac{\mathrm{d}P}{\mathrm{d}t} \right| \ll 1 ,$$

Eq. (2) implies (1). A natural generalization of Eq. (2) is to add a viscous friction term $\sim \nu\rho\, \Delta V$ into the right-hand side. Then in the quasi static approximation we arrive at the equation of Darcy–Brinkman [1]:

$$\nabla P = -\frac{\mu}{k} V + \nu\rho\, \Delta V . \tag{3}$$

In spite of the seemingly simple form of this equation the serious problem remains to determine the physical meaning of the coefficients in the Darcy–Brinkman law. The latter problem involves the analysis of microscopic hydrodynamics of fluid flow in pores. The simplest flow where Eq. (3) can be derived involves filtration through porous media with relatively simple structure, e.g., for granular media with low number density of stationary solid particles [4]. In this case ν is a regular kinematic viscosity ($\mu = \nu\rho$) and the coefficient k is determined by the Stokes law:

$$k = \frac{1}{\alpha d n} ,$$

where the coefficient α depends upon the shape of granules, d is the size of granules and n their number density. In case of spherical granules of diameter d the coefficient $\alpha = 3\pi$. For more realistic models of porous media like dense granular medium, ensemble of random capillaries etc. the coefficient ν does not equal the kinematic viscosity of fluid. Experiments show that the permeability k can be adequately described by the Carman–Kozeny relation [1]:

$$k = \frac{\varphi^3 a^2}{k_0 (1-\varphi)^2} ,$$

where the coefficient k_0 depends upon the structure of the porous medium ($k_0 \sim 45 - 10^3$), a is a characteristic dimension of grains of porous medium, φ is porosity.

The problem of determinating the coefficient in Brinkman's law is much more involved. Physical consideration show that viscous molecular effects in fluid filtration through the pores are accounted for by the local friction term in Darcy's law. However even in this case the Brinkman correction may be still valid only if it has a statistical rather than molecular origin. During filtration through an ensemble of random pore fluid acquires a random component of velocity $\boldsymbol{u}$ along with the regular one $\boldsymbol{V}$. It can be argued (see, e.g., [5]) that the average effect of a fluctuating component of velocity results in the Brinkman correction similarly to the statistical hydrodynamics [6] where the fluctuating component of velocity can be accounted for by turbulent viscosity in the averaged equation of motion.

In this study we analyze two-phase filtration of fluid through the stationary porous matrix or/and a two-phase flow. In the latter case instead of a porous matrix we consider the fluid with higher viscosity. It is showed that in the case of two-phase filtration, the perturbation of the phase composition during homogeneous background filtration of a two-phase mixture results in formation of structures similar to nondissipative shock waves [7] but not kinematic shock waves. Nondissipative shock wave emerges as a multi-soliton solution of the Korteweg–de Vries equation which describes the evolution of small perturbations of phase composition. Quite unexpectedly the dispersive properties of the Korteweg–de Vries equation are determined by the dissipative terms in the equations of motion, namely, by the coefficients near the second derivatives (Brinkman's correction).

2. Analysis of dynamic equations

Equations of two-phase filtration of incompressible fluids in the presence of gravity force read [8]:

$$\rho_i \Phi_i \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\Phi_i \nabla P + \rho_i \Phi_i \boldsymbol{g} - \frac{\mu_i}{k'} \Phi_i^2 \boldsymbol{v}_i + \operatorname{div} \hat{\sigma}_i - \frac{\mu_i}{K'} \Phi_i \Phi_j (\boldsymbol{v}_i - \boldsymbol{v}_j) , \tag{4}$$

$$\frac{\partial \Phi_i}{\partial t} + \nabla(\Phi_i \boldsymbol{v}_i) = 0 , \tag{5}$$

where ρ_i, Φ_i, $\boldsymbol{v}_i$ – density, volume fraction and velocity of the ith component, $\boldsymbol{g}$ – gravity acceleration, μ_{ij} some effective viscosity which determines the interphase friction between phases i and j, ξ_i, η_i – dynamic viscosity of the ith fluid (ξ_i is bulk viscosity), $\Phi_1 + \Phi_2 = 1$ and the tensor of viscous stresses in phase i is given by the following relation:

$$\hat{\sigma}_{km} = \Phi_i \eta_i \left(\frac{\partial v_k}{\partial x_m} + \frac{\partial v_m}{\partial x_k} - \tfrac{2}{3} \delta_{km} \frac{\partial v_n}{\partial x_n} \right) + \Phi_i \xi_i \delta_{km} \frac{\partial v_n}{\partial x_n} .$$

Coefficients k' and K' describe local friction between the porous matrix and filtrating fluids and friction between the two filtrating fluids (or equivalently, friction between filtrating fluid and viscous porous matrix), respectively. These coefficients can be determined from the Carman–Kozeny relation:

$$k' \equiv \frac{k}{\varphi} = \frac{a^2 \varphi^2}{k_0 (1-\varphi)^2} , \quad K' = \frac{b^2 \Phi^2 \varphi}{k_0 (1-\Phi)} ,$$

where $\Phi = \Phi_1$ is the volume fraction of the fluid 1 and φ is the porosity of the solid matrix. In case of fluid filtration through the viscous matrix coefficient b is a size of pores. Coefficient $\Phi_i \Phi_j \equiv \Phi(1-\Phi)$ in Eq. (4) is introduced for convenience. Note that although in the above relations the formula for K' is calculated from the expression for k' by substitution $\varphi \to \Phi\varphi$, the obtained results do not depend upon this particular choice of the model of two-phase filtration. In a general case the coefficient of interphase friction depends upon the relative velocity between phases. However since we consider only linear perturbations of the background uniform flow, this dependence results only in a numerical correction of the coefficients in dispersion relations derived below.

Note that without the solid matrix ($\varphi = 1$) the terms describing local friction between filtrating fluid and this matrix vanish: $k' = \infty$. Hereafter we assume that the multiplier μ in the term describing local friction between filtrating fluids in the case of filtration through viscoelastic matrix is equal to the dynamic viscosity of fluid. We restrict our analysis to the case of one-dimensional flow ($\boldsymbol{g}$ is parallel to x axis) and neglect the inertial terms. Then the system (4), (5) can be written as follows:

$$-\Phi \frac{\partial P}{\partial x} + \rho_1 \Phi g - \frac{\mu_1}{k'} \Phi^2 v + \frac{\partial}{\partial x} \left(\Phi \eta_1' \frac{\partial v}{\partial x} \right) - \frac{\mu}{K'} \Phi (1-\Phi)(v-u) = 0 , \tag{6}$$

$$-(1-\Phi) \frac{\partial P}{\partial x} + \rho_2 (1-\Phi) g - \frac{\mu_2}{k'} (1-\Phi)^2 u + \frac{\partial}{\partial x} \left((1-\Phi) \eta_2' \frac{\partial u}{\partial x} \right) + \frac{\mu}{K'} \Phi (1-\Phi)(v-u) = 0 , \tag{7}$$

$$\frac{\partial \Phi}{\partial t} = \frac{\partial [(1-\Phi) u]}{\partial x} , \tag{8}$$

$$\frac{\partial \Phi}{\partial t} = -\frac{\partial (\Phi v)}{\partial x} , \tag{9}$$

where $\eta' = \frac{4}{3}\eta + \xi$ and $v_1 \equiv v$, $v_2 \equiv u$. Since in this investigation we consider only the one-dimensional case the primes near the coefficients of viscosity are omitted.

The above equations (6), (7) describe two-phase filtration with a Darcy–Brinkman filtration law and with interphase friction proportional to the relative velocity between phases. Remarkably these equations can be derived from the stationary equations of conservation of momentum in multiphase flows where inertial effects are neglected and one of the phases is a rigid porous matrix. Therefore the main physical results of this investigation are relevant also in case of multiphase flows, e.g., flows of granular materials, fluidized beds etc.

3. McKenzie's equations

McKenzie [9] proposed a set of equations governing the filtration of viscous melt through very viscous partially molten deformable rock. This set of equations can be derived from the above system of equations (6), (7) when $\varphi = 1$ and $\eta_2' \gg \eta_1' \approx \mu$. Then Eqs. (6)–(9) yield:

$$-\Phi \frac{\partial P}{\partial x} + \rho_1 \Phi g - \frac{\mu}{K'(\Phi)} \Phi(1-\Phi)(v-u) = 0\,, \tag{10}$$

$$-(1-\Phi)\frac{\partial P}{\partial x} + \rho_2(1-\Phi)g + \frac{\partial}{\partial x}\left(\eta'(1-\Phi)\frac{\partial u}{\partial x}\right) + \frac{\mu}{K'(\Phi)}\Phi(1-\Phi)(v-u) = 0\,, \tag{11}$$

$$\frac{\partial \Phi}{\partial t} = \frac{\partial[(1-\Phi)u]}{\partial x}\,, \tag{12}$$

$$\frac{\partial \Phi}{\partial t} = -\frac{\partial(\Phi v)}{\partial x}\,. \tag{13}$$

Note that in the above equations the term which describes the interphase friction is written in symmetric form. Thus our notations are slightly different from those employed by McKenzie. Subtracting Eq. (13) from (12) we find that $\partial[\Phi v + (1-\Phi)u]/\partial x = 0$. Therefore the total flow rate $\Phi v + (1-\Phi)u = C(t)$ can only be a function of time. It is evident that in magma dynamics this integral equals zero since upward (downward) motion of a certain volume of magma is accompanied by downward (upward) motion of a mantle. The latter statement can be violated in the regions close to the Earth's surface where magma can percolate through the Earth's crust (e.g., vulcano's eruption).

Introducing the relative velocity of filtration $V = v - u$ we find:

$$v = V(1-\Phi)\,, \quad u = -V\Phi\,.$$

Substituting these relations into the system of equations (10), (11) yields:

$$(\rho_1 - \rho_2)g - \frac{1}{1-\Phi}\frac{\partial}{\partial x}\left(\eta(1-\Phi)\frac{\partial u}{\partial x}\right) + \frac{\mu}{K''}u = 0\,, \tag{14}$$

$$\frac{\partial \Phi}{\partial t} = \frac{\partial[(1-\Phi)u]}{\partial x}\,, \tag{15}$$

where

$$K'' = K'\Phi = \frac{\Phi^3}{(1-\Phi)}\frac{b^2}{k_0}\varphi\,.$$

The system of equations (14), (15) describes magma ascent through the porous mantle in the gravity field. In [9,10] it was shown that the system of equations (14), (15) describes propagation of nonlinear waves of phase composition (porosity) and velocity. In this study we demonstrate that for small deviations from the equilibrium state $\Phi_0 = \text{const.}$ $u_0 = (K''/\mu)g(\rho_2 - \rho_1)$ and the relatively small viscosity (parameter $\sqrt{(\eta'/\mu)K''}$ is much smaller than a integral length scale of the system) the system of equations (14), (15) (hereafter we call it McKenzie's system) implies the Korteweg–de Vries equation. We believe that the latter conclusion is of a very general nature: if inertia of filtrating fluid is neglected, quasi stationary two-phase filtration can be described by the Korteweg–de Vries equation for small deviations from the equilibrium. We will show also that this conclusion is valid also for filtration of two fluids through a rigid porous matrix. Therefore we consider a certain generalization of the McKenzie's system:

$$\frac{\partial \phi}{\partial t} + \frac{\partial(\phi u)}{\partial x} = 0\,, \tag{16}$$

$$\frac{1}{\phi}\frac{\partial}{\partial x}\left(\eta\phi\frac{\partial u}{\partial x}\right) = Q(u,\phi)\,, \tag{17}$$

where $\phi = 1 - \Phi$ is a volume fraction of a solid phase.

Eq. (17) describes an equilibrium between external (inviscid) forces (e.g., gravity force, interphase forces, external pressure) and viscous stresses induced in porous matrix. We assume that similarly to the McKenzie's system

$$Q_u \equiv \frac{\partial Q}{\partial u} > 0\,, \quad Q_\phi = \frac{\partial Q}{\partial \phi} < 0\,.$$

The first of the above conditions has a clear physical meaning. It means that the velocity dependent force $\propto Q$ describes the effect of local friction. The case $Q_u < 0$ corresponds to the unstable situation when initial perturbations will grow without any physical reason. Indeed transferring Q in Eq. (17) into the left-hand side and adding a term $\rho_2(\partial u/\partial t)$ into the right-hand side, it can be written as an equation of motion. Then condition $Q_u > 0$ means that the system evolves into the state which is described by Eq. (17), during a short time interval $\propto \rho_2/Q_u$ when it is disturbed from this state (compare with (2)). This time interval is very short indeed: assuming following McKenzie that $\mu \approx 10\,\mathrm{g/(cm\,s)}$, $b \approx 0.1\,\mathrm{cm}$, $\rho_2 \approx 3.3\,\mathrm{g/cm^3}$, $\phi \sim 10\%$, we find that $\rho_2/Q_u \sim 10^{-5}\,\mathrm{s}$. Certainly the real relaxation time will be larger since for such short processes the finite velocity of sound propagation must be taken into account. It must also be noted that many liquids (especially non-Newtonian) have finite internal relaxation time which is associated with molecular processes. Therefore it is not possible to consider phenomena with a characteristic time shorter than this relaxation time. In any case the characteristic times for filtration flows are much longer than any of the above mentioned ones. Evidently, the estimated time is negligibly small in comparison with any geophysical time and corresponds to the extremely high local friction. However as we will see further, the system as a whole exhibits the dispersive behavior. The sign in the condition $Q_\phi < 0$ is insignificant to some extent: it is associated with the choice of positive direction of the x axis in the direction of gravity acceleration.

Equilibrium (with constant velocity) flow u_0 is determined from the condition $Q(\phi_0, u_0) = 0$ or $u_0 = u_0(\phi_0)$. Note that in this investigation we consider the homogeneous equilibrium state since this case is the most interesting in various applications. Thus for every $\phi_0 < 1$ there exists an equilibrium flow. Linearizing the system of equations (16), (17) near equilibrium and considering the solution $\sim\exp(-\mathrm{i}\omega t + \mathrm{i}kx)$ we find the following dispersion equation:

$$\omega = ku_0 + \frac{\tilde{u}k}{1 + D^2k^2}\,, \tag{18}$$

where

$$\tilde{u} = -\phi_0\frac{Q_\phi}{Q_u} = \phi_0\frac{\partial u_0(\phi_0)}{\partial \phi_0}$$

is a characteristic velocity (derivatives Q_u and Q_ϕ are calculated at an equilibrium state ϕ_0, u_0) and

$$D = \sqrt{\frac{\eta}{Q_u}} \approx \sqrt{\frac{\eta}{\mu}\frac{K'}{\phi(1-\phi)}} \tag{19}$$

is the internal characteristic length scale of the problem (dispersive length). Note that in the geophysical applications this quantity is called a compaction length. Adopting according to [10] $\eta \approx 10^{16}\,\mathrm{cm/s}$, we find that $D \sim b\sqrt{1-\phi} \times \sqrt{\eta/\mu} \approx 1\,\mathrm{km}$. This dispersive length is small in comparison with the integral length scale of the system (several hundred kilometers). Note that when $u_0 \to 0$, the dispersive equation (18) is similar to the dispersive relations encountered for slow Rossby waves in a rotating medium. A typical dispersion curve is shown in Fig. 1. Note that in the above linear analysis the dispersion is not assumed to be small.

This dispersion equation clearly exhibits cubic dispersion law. Indeed, for $kD \ll 1$ (long wave

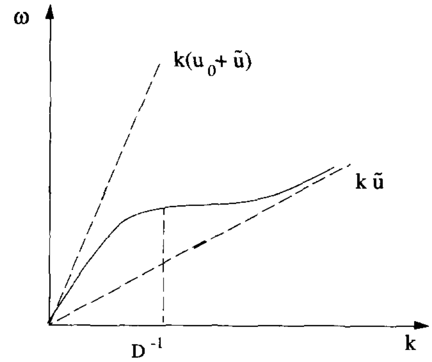

Fig. 1. Dispersion curve of the generalized McKenzie system (16), (17).

approximation), $\omega \cong k(u_0 + \tilde{u}) - \tilde{u}k^3D^2$. It can easily be seen that in this system dispersion is related directly to the viscosity of porous matrix (Earth's mantle): $D^2 \propto \eta$. The latter conclusion, unexpected as it seems, has a very clear physical meaning: in porous medium filtration is feasible only if external forces (e.g., gravity or external pressure gradient) perform continuously work at filtrating fluid. Without this work filtration stops after a very short time interval $\propto \rho/Q_u$. In such systems filtration is supported by a continuous supply of energy and the rate of energy supply equals to dissipation rate.

It is known that when viscous effects are taken into account in acoustics, they result in damping of acoustic oscillations. However, damping is also accompanied by a small frequency shift. It can be shown that in this case also $\Delta\omega = -(\eta/\rho) \times k^3/8c_s$, where c_s is the velocity of sound. Indeed, the equation for sound waves propagation with dissipation reads:

$$\frac{\partial^2 \rho}{\partial t^2} = c_s^2 \nabla^2 \rho - \frac{\eta}{\rho_0} \nabla^2 \frac{\partial \rho}{\partial t},$$

where $\rho = \Delta\rho/\rho_0$ is the relative density perturbation. This equation yields the following dispersion relation:

$$\omega^2 = k^2 c_s^2 - \mathrm{i}\omega \frac{\eta}{\rho} k^2 .$$

Assuming that $kc_s \gg (\eta/\rho_0)k^2$ we obtain that $\mathrm{Re}\{\omega\} \approx kc_s - (\eta/\rho_0)^2(k^3/8c_s)$ and $\mathrm{Im}\{\omega\} \approx -\frac{1}{2}(\eta/\rho_0)k^2$. Thus the term which is responsible for dissipation causes also cubic dispersion along with wave damping. Imagine that energy is pumped into such a system until the onset of auto oscillations. In this case the effects of dissipation are insignificant since in such an open system the energy input compensates energy losses, but dispersion still remains. The similar effect occurs in filtration flows where additional losses caused by nonlocal friction $\eta\,\Delta u$ are negligibly small in comparison with local friction losses.

Now we derive the Korteweg–de Vries equation from the system of equations (16), (17). Consider small (but finite) perturbation of the equilibrium flow ϕ_0, u_0:

$$\begin{aligned} &\phi = \phi_0 + \phi_1 , \quad |\phi g_1| \ll \phi_0 , \\ &u = u_0 + u_1 , \quad |u_1| \ll u_0 , \\ &\frac{\phi_1}{\phi_0} \sim \frac{u_1}{u_0} \sim \varepsilon \ll 1 . \end{aligned} \tag{20}$$

Consider also a long-wave approximation (with weak dispersion) case, i.e., $\eta \ll 1$ and therefore

$$kD \propto \sqrt{\varepsilon} \ll 1 . \tag{21}$$

Substituting (20) into the system of equations (16), (17) and keeping only terms of order lower or equal to ε^2 we arrive at the following system of equations:

$$\frac{\partial \phi_1}{\partial t} + \phi_0 \frac{\partial u_1}{\partial x} + u_0 \frac{\partial \phi_1}{\partial x} + \frac{\partial(\phi_1 u_1)}{\partial x} = 0 , \tag{22}$$

$$\begin{aligned} &\eta \frac{\partial^2 u_1}{\partial x^2} - Q_\phi \phi_1 - Q_u u_1 - Q_{\phi\phi} \cdot \tfrac{1}{2}\phi_1^2 \\ &\quad - Q_{\phi u}\phi_1 u_1 - Q_{uu} \cdot \tfrac{1}{2} u_1^2 = 0 . \end{aligned} \tag{23}$$

Solving Eq. (23) for u_1 we find that

$$\begin{aligned} u_1 = &- \frac{Q_\phi}{Q_u} \phi_1 \\ &+ \frac{1}{Q_u} \left(\eta \frac{\partial^2 u_1}{\partial x^2} - \tfrac{1}{2}\phi_1^2 Q_{\phi\phi} - Q_{\phi u}\phi_1 u_1 \right. \\ &\left. - Q_{uu} \cdot \tfrac{1}{2} u_1^2 \right) . \end{aligned} \tag{23'}$$

Then solving the latter equation (23′) by iterations and substituting the result into Eq. (22) (with an accuracy of order ε^2) we find:

$$u_1 = u_1^{(1)} + u_1^{(2)} + \cdots .$$

After the first iteration of Eq. (23′) we find:

$$u_1^{(1)} = -\frac{Q_\phi}{Q_u}\phi_1 \propto \varepsilon .$$

The second iteration yields:

$$u_1^{(2)} = -\frac{1}{Q_u}\left[\eta \frac{Q_\phi}{Q_u}\frac{\partial^2\phi_1}{\partial x^2} + \frac{\phi_1^2}{2}\left(Q_{\phi\phi} - 2\frac{Q_{\phi u}Q_\phi}{Q_u} + \frac{Q_\phi^2 Q_{uu}}{Q_u^2}\right)\right] \propto \varepsilon^2 .$$

Note that during the solution of Eq. (23′) by iterations we explicitly used the fact that $\eta \ll 1$ and therefore $\eta\phi_1 \sim \varepsilon^2$. Substituting the determined solution for u_1 into the equation for ϕ_1 (22) we finally find:

$$\frac{\partial\phi_1}{\partial t} + (u_0 + \tilde{u})\frac{\partial\phi_1}{\partial x} + \frac{2}{\phi_0}(\tilde{u} + \tilde{\tilde{u}})\phi_1\frac{\partial\phi_1}{\partial x} - \tilde{u}D^2\frac{\partial^3\phi_1}{\partial x^3} = 0 , \tag{24}$$

where

$$\tilde{\tilde{u}} = -\left(\frac{1}{2}\frac{Q_{\phi\phi}}{Q_u} - \frac{Q_\phi Q_{\phi u}}{Q_u^2} + \frac{1}{2}\frac{Q_\phi^2 Q_{uu}}{Q_u^3}\right)\phi_0^2 .$$

Introducing the new variables $t' = t$, $x' = x - (u_0 + \tilde{u})t$ (writing Eq. (23) in a moving frame) and rescaling the spatial variable we arrive at the Korteweg–de Vries equation:

$$\frac{\partial\phi_1}{\partial t} + \phi_1\frac{\partial\phi_1}{\partial x} \pm \beta^2\frac{\partial^3\phi_1}{\partial x^3} = 0 , \tag{25}$$

where the plus and minus signs correspond to $\tilde{u} + \tilde{\tilde{u}} < 0$ and $\tilde{u} + \tilde{\tilde{u}} > 0$, respectively. The normalized velocity $u_1/\tilde{u}$ satisfies the same Korteweg–de Vries equation (24). Note that the Korteweg–de Vries equation (25) is valid only in a small (but finite) amplitude and long-wave approximation while the dispersion relation (18) is valid for arbitrary η.

Occurrence of the Korteweg–de Vries equation in the system of equations describing the viscous flow seems rather unexpected at the first sight since the Korteweg–de Vries equation describes dissipationless processes. However as it was discussed above in the McKenzie system dissipation is compensated by external pumping of energy (gravitation and external pressure gradient). Note that in the analysis of magma's filtration the effects of phase transitions can play an important role. Here we adopt the simple model of filtration and do not take these effects into account.

4. Formation of nondissipative shock waves

As was demonstrated above, nonlinear waves with sufficiently small amplitudes in viscoelastic medium are described by the Korteweg–de Vries equation. Therefore, all the phenomena that are encountered in the systems which are described by the Korteweg–de Vries equation, like solitons, periodic waves, dissipationless shock waves can occur also in viscoelastic medium. In the following we analyze the phenomenon of formation of dissipationless shock waves.

Consider an initial value problem for Korteweg–de Vries equation (25) with smooth initial data (see Fig. 2a):

$$u(x,0) = u_0(x) > 0 , \quad \int_{-\infty}^{+\infty} u_0(x)\,\mathrm{d}x = 1 . \tag{26}$$

It is known that for sufficiently large time $t \geq \beta^{-1}$, such initial distribution evolves into a chain of N solitons ordered according to their amplitude (see Fig. 2b), where $N \sim 1/\pi\beta\sqrt{\varepsilon}$ for $\beta \ll 1$ [11]. Qualitatively the evolution of the initial distribution is showed in Fig. 2. So long as the solution $u(x,t)$ remains smooth, the evolu-

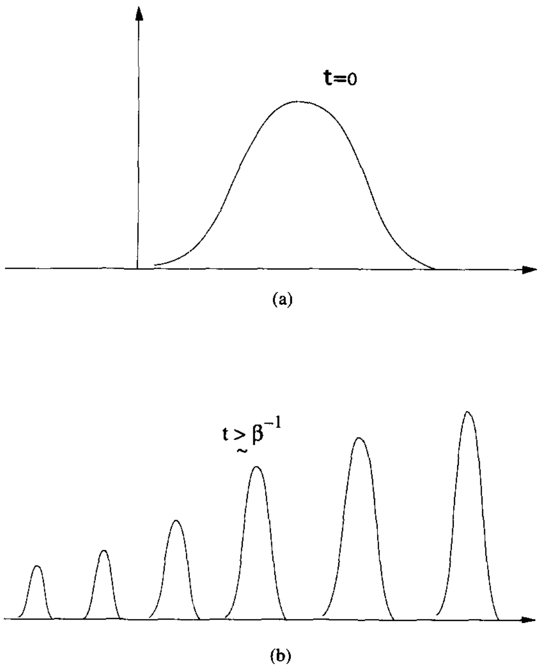

Fig. 2. (a) Initial data for the KdV equation (24); (b) *N*-soliton chain.

tion is governed by Eq. (25) with $\beta = 0$. In this case the solution can be found explicitly:

$$u = u_0(x - ut)\,, \quad x = ut + u_0^{-1}(u)\,. \tag{27}$$

Obviously this solution will break and become multi valued after finite time $t = t_{br}$ at $x = x_{br}$ (see Fig. 3b). In the vicinity of $t = t_{br}$; $x = x_{br}$ the term $\beta^2(\partial^3 u/\partial x^3)$ in Eq. (24) cannot be neglected and function (27) ceases to be a solution. It can be showed that for $t > t_{br}$ in the vicinity of $x = x_{br}$ a domain in the plain (x, t) is formed where the solution exhibits high frequency oscillations (see Fig. 3c and d). Thus after $t > t_{br}$ a continuously expanding domain filled with high frequency oscillations is formed. This domain is called a nondissipative shock wave (NDSW) since it is similar to the regular shock waves in gas dynamics or nonlinear elasticity which are formed in the media with high dissipation in the regions with high gradients. Oscillations inside a nondissipative shock wave can be described by modulated cnoidal (elliptic) Jacobi functions [7]:

$$u(x, t) = a\,\mathrm{Cn}\Big(\frac{x - Ut}{\beta}\,k\,\Big|\,m\Big) + b\,, \tag{28}$$

where the amplitude a, wave number k, parameter of ellipticity m and phase velocity U are smooth functions of x and t. Since x and t appear in these functions as x/ε, t/ε we assert that these functions vary slowly in comparison with oscillations with frequency $\propto 1/\beta$. The mathematical theory of the nondissipative shock waves was pioneered by Whitham [7] and the Riemann problem for NDSW was solved first in [12]. Qualitatively this solution of Riemann's problem for NDSW is shown in Fig. 4. The full analytical solution for nondissipative shock wave with smooth unimodal initial distribution (explicit formulas for leading $x_+(t)$ and trailing $x_-(t)$ fronts, formulas for smooth functions $a(x, t)$, $m(x, t)$ etc.) was found in [13] (see Fig. 5). It is of interest to note that similar solutions were apparently detected in recent numerical experiments [14,15].

Consider now the physical meaning of the NDSW in multiphase hydrodynamics. It is quite clear that in the case of multiphase filtration formation of NDSW corresponds to formation of inhomogeneities of phase composition. Kinematic shock waves result in formation of macroscopic inclusions of one phase inside the other phase with characteristic size of order of the dispersive length. The latter phenomenon is most likely typical for two-phase filtration and may represent the universal mechanism of a number of experimentally observed effects: formation of water fingers inside oil during oil displacement by water in oil recovery process [1,5]; formation of magma inclusions in earth's mantle and crust [3,9,10]; formation of gas bubbles in a fluidized bed of granular material [16,17].

Note that although the solitary-wave solutions of McKenzie's equations were discovered in previous investigations (see, e.g., [3,10,18,19]) to the best of our knowledge the phenomena associated with formation of the dissipationless

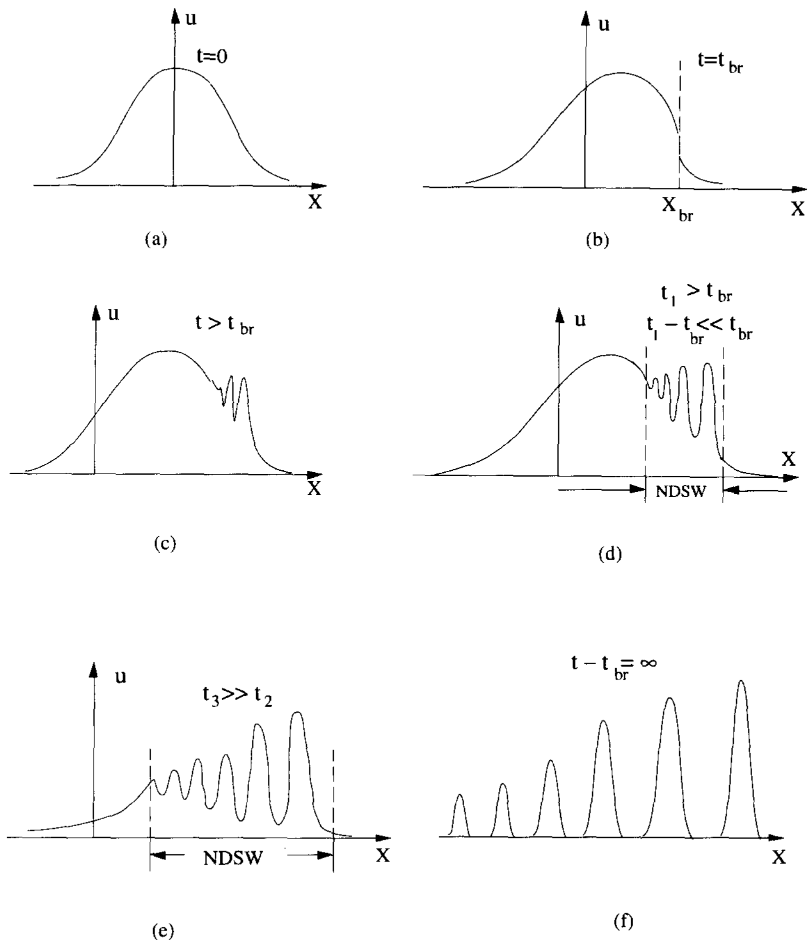

Fig. 3. Stages of evolution of smooth initial distribution and formation of NDSW.

shock waves in filtration flows were not studied before.

It is essential that the above described situation occurs only if coefficient of dynamic viscosity in Darcy's law is significantly less than the coefficient in the Brinkman's term. Only in this case $D \gg b$ and the formed inhomogeneities are of macroscopic size and can be observed in experiments. Except for the special case of magma filtration through very viscous partially molten rock, the required relation between dynamic viscosity and Brinkman's coefficient can be satisfied if the Brinkman's term is not of molecular origin. Thus, e.g., in a fluidized bed of granular material this terms arises due to collisions between particles of granular material suspended in a gas stream.

5. Two-phase filtration through rigid porous matrix

Consider a more general case of filtration flows which are described by system of equations (6)–(9) when the effect of the solid porous matrix is taken into account. This problem is of relevance in many technological applications, like oil reservoir engineering, catalytic bed and fluidized bed reactors in chemical engineering.

Subtracting Eq. (9) from (8) we find that the

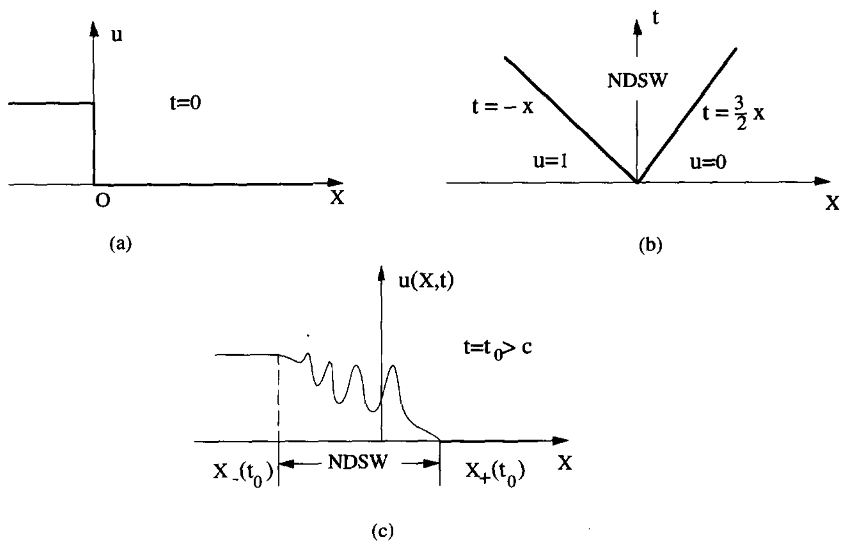

Fig. 4. Solution of the Riemann problem for NDSW.

total flow rate $W = v\Phi + (1-\Phi)u$ is a spatial integral of motion. However in this case the total flow rate does not equal zero. Introducing the relative velocity $V = u - v$ we find:

$$v = W + V(1-\Phi)\,, \quad u = W - \Phi V\,.$$

Substituting the latter expressions into (6)–(9) and excluding pressure yields:

$$(\rho_1 - \rho_2)g + (\Phi\mu_0 - \mu_1)W + \eta_+ \frac{\partial^2 \xi}{\partial x^2} + \xi\eta_- \frac{\partial^2 \Phi}{\partial x^2} + \tilde{\eta}_- \frac{\partial \xi}{\partial x}\frac{\partial \Phi}{\partial x} + \xi\tilde{\eta}_+ \left(\frac{\partial \Phi}{\partial x}\right)^2 - \xi\left[\frac{\mu_0}{k'} + \frac{\mu}{\bar{K}}\right] = 0\,, \tag{29}$$

$$\frac{\partial \Phi}{\partial t} + W\frac{\partial \Phi}{\partial x} + \frac{\partial \xi}{\partial x} = 0\,, \tag{30}$$

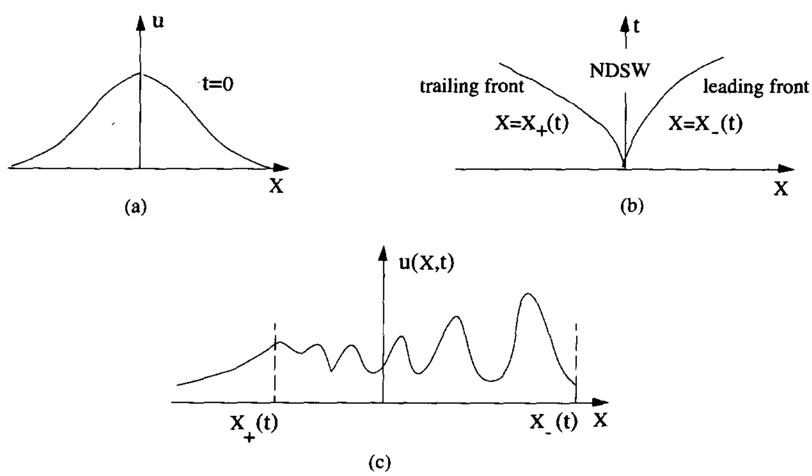

Fig. 5. Behavior of NDSW evolving from unimodal initial distribution.

where

$$\mu_0 = \mu_1 + \mu_2 \quad \xi = \Phi(1-\Phi)V\,,$$
$$\bar{K} = K'\Phi(1-\Phi)\,,$$
$$\eta_+ = \frac{\eta_1}{1-\Phi} + \frac{\eta_2}{\Phi}\,, \quad \eta_- = \frac{\eta_1}{(1-\Phi)^2} - \frac{\eta_2}{\Phi^2}\,,$$
$$\tilde{\eta}_- = \frac{\eta_1}{1-\Phi} - \frac{\eta_2}{\Phi}\,, \quad \tilde{\eta}_+ = \frac{\eta_1}{(1-\Phi)^2} + \frac{\eta_2}{\Phi^2}\,,$$
$$\mu_{12} \equiv \mu\,.$$

Note that the total flow rate W appears explicitly in the system of equations (29), (30) and is determined by the pressure gradient in the system. Adding Eqs. (6) and (7) we find:

$$\begin{aligned}\frac{\partial P}{\partial x} &= [\rho_1\Phi + \rho_2(1-\Phi)]g \\ &\quad - \frac{W}{k}[\mu_1\Phi^2 + \mu_2(1-\Phi)^2] - \xi\frac{\mu}{k'} \\ &\quad - \eta_2\frac{\partial}{\partial x}\left[(1-\Phi)\frac{\partial}{\partial x}\left(\frac{\xi}{1-\Phi}\right)\right] \\ &\quad + \eta_1\frac{\partial}{\partial x}\left[\Phi\frac{\partial}{\partial x}\left(\frac{\xi}{\Phi}\right)\right]. \end{aligned} \tag{31}$$

Assume that all the values are constant at infinity and that the deviation of the solution from these constants behaves like a localized perturbation. Then Eq. (31) yields:

$$\begin{aligned}\lim_{\Delta L\to\infty}\frac{\Delta P}{\Delta L} &= \{\rho_1\Phi + \rho_2(1-\Phi)\}g \\ &\quad + \{\mu_1\Phi^2 + \mu_2(1-\Phi)^2\}\frac{W}{k} + \frac{\mu}{k'}\{\xi\}\,, \end{aligned} \tag{32}$$

where $\{A\} = \frac{1}{2}[A(\infty) + A(-\infty)]$.

Note that relations of this type are encountered frequently in hydraulics. The system of equations (29), (30) can be generalized similarly to McKenzie's systems:

$$\begin{aligned}&\eta_+\frac{\partial^2\xi}{\partial x^2} + \xi\eta_-\frac{\partial^2\phi}{\partial x^2} \\ &\quad = Q(\xi,\phi) + \tilde{\eta}_-\frac{\partial\xi}{\partial x}\frac{\partial\phi}{\partial x} - \eta_+\xi\left(\frac{\partial\phi}{\partial x}\right)^2, \end{aligned} \tag{33}$$

$$\frac{\partial\phi}{\partial t} + W\frac{\partial\phi}{\partial x} + \frac{\partial\xi}{\partial x} = 0\,. \tag{34}$$

However this generalization does not have such clear meaning as the generalized McKenzie system. In this case it is not possible to make an unambiguous statement about the sign of Q_ϕ. Still the claim that $Q_\xi > 0$ is valid since it is related to the stability of the system and guarantees that the corresponding linear operator is positive definite. An equilibrium flow is determined by the same equations as before:

$$\phi_0 = \text{const.}\,, \quad Q(\phi_0, \xi_0) = 0\,.$$

Linearizing the system of equations (33), (34) in the vicinity of this equilibrium state after some algebra we determine the following dispersion equation:

$$\omega = k(W + \tilde{u}) + \frac{V_0\bar{D}^2\,\mathrm{sgn}(Sk^3)}{1 + D_+^2k^2}\,, \tag{35}$$

where

$$\tilde{u} = V_0(1-2\phi_0) + \phi_0(1-\phi_0)\frac{\partial V_0}{\partial\phi_0}\,, \tag{36}$$

and

$$S = \Delta\eta + \eta_0\frac{\partial\ln V_0}{\partial\phi}\,, \quad \Delta\eta = \eta_2 - \eta_1\,,$$
$$\eta_0 = \eta_1\phi + \eta_2(1-\phi)\,, \quad D_+ = \sqrt{\frac{\eta_+}{Q_{\xi_0}}}\,.$$

The characteristic dispersion length scale is

$$\bar{D} = \sqrt{\frac{|S|}{Q_{\xi_0}}}\,. \tag{37}$$

In order to derive the above relations we used the identity which follows from the definition of ξ_0 and Q. Indeed, since in equilibrium the total derivative of Q vanishes, e.g., $(\mathrm{d}Q/\mathrm{d}\phi)_{\xi_0(\phi_0)} = 0$, we find that

$$\begin{aligned}\left.\frac{\partial Q}{\partial\phi}\right|_{\phi_0,V_0} &\equiv -\left.\frac{\partial Q}{\partial\xi}\right|_{\phi_0,V_0}\left(\phi_0(1-\phi_0)\frac{\partial V_0}{\partial\phi_0}\right. \\ &\quad \left. + (1-2\phi_0)V_0\right)\end{aligned}$$

and then the dispersion relation (35) and the expression for the dispersion length scale (37) are immediately obtained.

Since $\eta_0 < \max\{\eta_1, \eta_2\}$ and $\eta_+ > \max\{\eta_1, \eta_2\}$, generally $D_+ > \bar{D}$ (except for the case $\mathrm{d}\ln V_0/\mathrm{d}\phi \gg 1$). Therefore these two scales determine the behavior of the dispersion curve which can be rather involved (see, e.g., Fig. 6).

Note that in a case pertinent to fluidized bed when velocity of one of the phases is zero, $W = V_0(1-\phi)$ and according to (36) the velocity of propagation of a linear disturbance is given by the following relation:

$$u^* = W + \tilde{u}$$
$$= 2V_0\left[1 - \tfrac{3}{2}\phi_0\left(1 - \tfrac{1}{3}(1-\phi_0)\frac{\partial \ln|V_0|}{\partial \phi_0}\right)\right]. \tag{38}$$

It is easily seen that when dynamic viscosities of both fluids are equal the derived dispersion equation coincides with that obtained for McKenzie's system. However a wide variety of dispersion equations can occur in this case (see, e.g., Fig. 6). Similarly to the generalized McKenzie's system, the Korteweg–de Vries equation can also be derived from the system of equations (33), (34): it coincides with (24) with the substitution $\tilde{u}D^2 \to V_0\bar{D}^2 \operatorname{sgn} S$, $Q \to Q\phi$, $u \to \xi$.

Approximation of weak dispersion is valid when the characteristic dispersion length scale is small. However an excessively small dispersion length is also unacceptable since it cannot be smaller than the pore size. Note that if

$$(\rho_1 - \rho_2)g - \frac{1}{1-\Phi}\frac{\partial}{\partial x}\left(\eta(1-\Phi)\frac{\partial u}{\partial x}\right) + \frac{\mu}{K''}u = 0$$

this dispersion length is always smaller or of the same order as the pore size. Therefore the necessary condition of validity of the above approach is $(\eta/\mu)b^2 \gg b^2$. Obviously the latter condition implies $\mu \ll \eta$. In dense packed beds the condition $\mu \ll \eta$ is feasible only if η is not of molecular origin. Estimations of characteristic values show that $\eta \sim V_0^2(k\rho^2/\mu)$. Assuming that $V_0 \sim (\Delta P(k/\Delta L)\mu)$ we find that $\eta/\mu \sim (\Delta P/\Delta L)^3(k^3\rho^2/\mu^4)$. Therefore in the systems with relatively fast filtration and low viscosities of filtrating fluids such effect cannot be ruled out.

Fluidized bed of granular material provides another example of two-phase medium with high value of effective viscosity coefficient. In the latter case high viscosity arises due to collisions between particles of granular material. Using expression (37) to estimate the characteristic size of gas bubbles R^* which are formed in the fluidized bed we find that since $V_0 = -\rho g K'/\mu$, $\partial \ln|V_0|/\partial\phi = \partial \ln K'/\partial\phi$ and:

$$R^* \sim b\sqrt{\frac{\rho_s n \cdot \frac{1}{6}\pi b^3 \nu_s f(\phi)|1 + \partial \ln K'/\partial\phi|}{\rho_f \nu_f \phi(1-\phi)}}, \tag{39}$$

where ν_s is an effective kinematic viscosity of solid phase, n is a number density of solid particles, b is a diameter of solid particles, ρ_s and ρ_f densities of solid particles and gas, respectively. In relation (39) we used the experimental correlation presented in [16] whereby $K' = \frac{2}{9}b^2 f(\phi)$ where $f(\phi) = \exp[(0.3652\phi - 4.093)^{1/3}\sqrt{(1-\phi)/\phi}]$. Substituting the characteristic values of the parameters of fluidized bed ($\bar{U} \sim 1\,\mathrm{m/s}$, $\rho_s/\rho_f \sim 10^3$, $b \sim 0.5\,\mathrm{mm}$, $\phi \sim 0.5$) and using elementary kinetic theory estimate for ν_s whereby $\nu_s \propto V_0/\pi b^2 n$ we find that $R^* \sim 0.05$–$0.10\,\mathrm{m}$ which is in good agreement with experimental observations and with the results of one-dimensional numerical simulations [17]. The

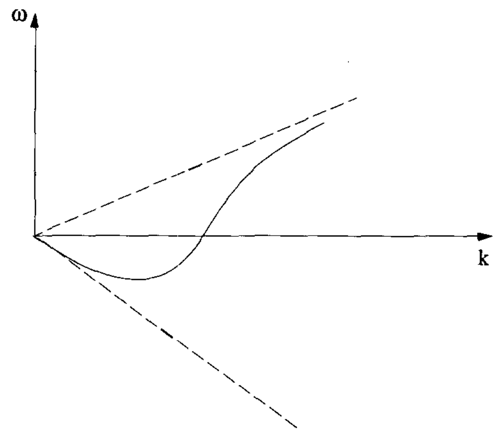

Fig. 6. Dispersion curve for the system of equations (32), (33).

velocity of bubbles can be estimated as a phase velocity u^* and it is of order $1.96V_0$ for the same parameters of fluidized bed, i.e., bubbles move faster than gaseous phase in compliance with experiments.

6. Conclusions

In this investigation we analyzed the evolution of perturbations of phase composition and relative velocity of phases in two-phase flows, e.g., multiphase filtration in porous media. In the developed theory we took into account the effects of local friction, Stokes type friction, gravitation effects but neglected the inertial terms. The principal result is that the evolution of perturbation of small but finite amplitude can be described by the Korteweg–de Vries equation which is encountered in various physical and technological problems. Similar approach was developed in [20] taking into account the inertial effects. In turns out that when inertial effects are taken into account the evolution of a small amplitude perturbation can be described by the Korteweg–de Vries–Burgers equation. In [21] it was demonstrated that conservation equations of two-phase hydrodynamics admit periodic solutions. However the estimations show that generally the contribution of inertial effects is rather small and becomes significant only at very long times. The suggested above scenario of evolution of the perturbations as the non stationary dissipationless shock waves describes in fact the transitional regime between the initial perturbation and stationary traveling wave solution of the Korteweg–de Vries–Burgers equation. The latter statement is certainly valid if the solution is stable. The unstable case requires a special analysis and will be considered elsewhere. It is well known that the soliton-type solutions of multi-dimensional Korteweg–de Vries equation are unstable. Probably this is the reason for instability of the analyzed above patterns in real flows. In conclusion it is worthwhile noting that a comparatively simple model developed in this investigation allows us to predict correctly a variety of two-phase flows which are important in view of their applications: two-phase filtration and fluidized bed.

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