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AN EVOLUTION MODEL FOR MONTE CARLO ESTIMATION OF EQUILIBRIUM NETWORK RENEWAL PARAMETERS

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This paper presents Monte Carlo techniques for evaluating equilibrium availability and mean up and down periods of a renewable network for a wide class of network operational criteria. The suggested method is based on a graph evolution model that overcomes the main difficulty—hitting low-probability "border" states of the criterion. Theoretical efficiency of the method is briefly discussed and numerical results are presented.

1. INTRODUCTION

By a network we mean a pair $N = (G, UP)$ consisting of the following:

An undirected graph $G = (V, E)$ with the node-set V and edge-set E; subsets of edges are sometimes called states of the network.

A class UP of operational (or "up") states $F \subseteq E$; the complementary class of "down" states is denoted by DN. A partition of the state space into UP and DN is sometimes referred to as the network operational criterion. We always assume this criterion to be *monotonic*: $F \in UP$ and $F \subseteq F'$ imply $F' \in UP$.

In this paper we confine ourselves to operational criteria admitting formulation in terms of node-sets of components of the subgraph (V, F) (including

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one-node components, if any). The partition of V into the components of (V, F) will be denoted by $\langle F \rangle$. Formally, we require that

$$F \in UP \quad \text{and} \quad \langle F \rangle = \langle F' \rangle \quad \text{imply } F' \in UP. \tag{1.1}$$

In applied random graphs theory, a network reliability problem is posed in the following "static" form (see, e.g., Colbourn [2]):

Given a function $q: E \to [0,1]$ (edge failure probabilities), define state probabilities by

$$P(F, q) = \prod_{e \in F} p(e) \prod_{e \in E-F} q(e), \tag{1.2}$$

where $p(e) = 1 - q(e)$ is the up probability of an edge. What is then

$$R(G,q) = \sum_{F \in UP} P(F,q) \quad (\text{or } Q(G,q) = \sum_{F \in DN} P(F,q))? \tag{1.3}$$

In application to renewable networks, Eq. (1.3) expresses the equilibrium instant availability of the network. Namely, suppose that the life of an edge e is described by an alternating renewal process $\kappa_e(t)$ taking the values *up* and *down*. Suppose that the edge processes are independent. We assume only that there exist the mean up and down periods for each $\kappa_e(t)$, $T_{UP}(e)$ and $T_{DN}(e)$, and consequently the failure rate $\lambda(e) = [T_{UP}(e)]^{-1}$ and the repair rate $\mu(e) = [T_{DN}(e)]^{-1}$ (see Gnedenko [6, p. 111]). For each $\kappa_e(t)$ it is assumed that the up and down epochs have nonlattice distributions. Consider the process $E(t) = \{e \in E: \kappa_e(t) = up\}$. The limit probability (as $t \to \infty$) that $E(t) \in UP$ at a randomly chosen moment t equals $R(G,q)$, with q given by

$$q(e) = \frac{\lambda(e)}{\lambda(e) + \mu(e)}. \tag{1.4}$$

Another interesting characteristic of a renewable system is the so-called stationary transition rate [6]:

$$\Phi(G,q) = \sum_{F \in DN} P(F,q) \cdot \mu(J_F), \tag{1.5}$$

where J_F is the set of edges $e \notin F$ such that $F + e \in UP$, and $\mu(J_F) = \sum\{\mu(e): e \in J_F\}$ (see also Keilson [7] for a purely Markov case).

The parameters $R(G,q)$ and $\Phi(G,q)$, interesting for their own sake, can be used for calculating the mean equilibrium up and down periods of the network. These periods, denoted by T_{UP} and T_{DN}, are defined as follows. Consider the sequence $U_1, U_2, \ldots, U_n$ of successive up periods of the network (as a whole) and the sequence $D_1, D_2, \ldots, D_n$ of its successive down periods. Then, as $n \to \infty$, $U_n \overset{Pr}{\to} U$, $D_n \overset{Pr}{\to} D$ (see Gnedenko [6, pp. 79, 111]), and $T_{UP} = \mathbb{E}[U]$, $T_{DN} = \mathbb{E}[D]$. Then, the following formulas are valid (see Gnedenko [6, p. 111]):

$$R(G,q) = \frac{T_{UP}}{T_{UP} + T_{DN}}, \qquad \Phi(G,q) = \frac{1}{T_{UP} + T_{DN}}. \tag{1.6}$$

The preceding model may have various applications. One of them is the reliability analysis of systems represented by undirected graphs or, more generally, by matroids. Indeed, the only thing needed to make the model serve for the arbitrary matroid defined on E is to interpret the symbol $\langle F \rangle$ (see Section 3 later) as a closure of a subset F in the given matroid.

Another application is availability analysis of a communication network. Such a network N, represented in our case by an undirected graph G and a capacity function $w(e)$, is at any moment engaged in performing a number of tasks. Suppose that N is in a stationary regime (equilibrium) and a new job J enters. This may be a communication demand for two or more nodes (terminals). Depending on the current situation in the network, only a part of links (edges) may be available for use by J. The states of N can be partitioned into the classes UP and DN with respect to J, and one may ask what is the probability that J finds N in the state UP and, if not, what is the expected waiting time for J to be served by N.

There are various approaches to the static network reliability problem. Precisely computing R is, in general, #P-hard, even for such simple criterion as the overall connectivity [12]. Valuable information about network reliability, qualitative as well as numerical, comes from various bounds based on a few relevant polynomial-time invariants of a graph, such as the size of mincut, the number of mincuts and spanning trees [1,8,9,10], and maximal spanning trees packing [11].

In practice, Monte Carlo (MC) opens a direct way to compute network reliability for any operational criteria, and investigation of MC methods should take its place in applications of random graphs theory. By MC evaluation of a reliability parameter Z, we mean constructing a random variable $Z(u)$ on some finite probabilistic space (U, π) (an "urn" U with choice probabilities $\pi(u)$, $u \in U$) with the property that an average

$$\hat{Z}_N = \frac{1}{N} \sum_{i=1}^{N} z(u_i) \tag{1.7}$$

over an independent sample $u_1, \ldots, u_N$ from U tends to Z with probability 1 when $N \to \infty$. We always assume that both generating u with distribution $p(u)$ and computing $Z(u)$ for a given $e \in U$ are polynomial-time (in the number of nodes).

Usually, the edge failure probabilities depend on a parameter that makes the q-vector vary from all ones (absolutely unreliable network) to all zeroes (absolutely reliable network). We adopt such parametrization in the form

$$q(e) = \exp[-w(e) \cdot t], \qquad 0 \le t < \infty, \tag{1.8}$$

with nonnegative "capacities" $w(e)$, $e \in E$. One may understand $w(e)$ as "thickness" or a number of independent standard "wires" forming an edge e, each wire having failure probability e^{-t}. Furthermore, one may not have to specify a graph G but rather consider w as defined on the edge-set of the complete graph and assign zero capacities to edges from outside E. So we use notations $Z(G, q)$ and $Z(w, t)$ as synonyms, for any reliability measure Z.

DEFINITION: *A* MC *scheme* $(U, \pi, Z(t \mid u))$ *for evaluating a reliability measure* $Z(w, t)$ *is called homogeneous with respect to a given parameter* t *(introduced by Eq.* (1.8) *or in another way) if the choice probabilities* $\pi(u)$ *are independent of* t.

A basic example of a nonhomogeneous scheme is the so-called crude Monte Carlo (CMC): U is the set of states (i.e., all subsets F of E), the choice probabilities $\pi(F)$ are given by Eq. (1.2) and, say, $Z(F) = 1$ for $F \in UP$ and 0 for $F \in DN$.

Throughout this paper we confine ourselves to unbiased MC schemes:

$$\mathbb{E}[Z(t \mid \cdot)] = \sum_{u \in U} \pi(u) \cdot Z(t \mid u) = Z(w, t) \tag{1.9}$$

and use the square relative error (s.r.e.)

$$\delta_Z^2(w, t) = \frac{\operatorname{var}[Z(t \mid \cdot)]}{Z(w, t)^2} \tag{1.10}$$

as an efficiency measure of the scheme. Homogeneous schemes are good from at least two points of view. First, a run of such scheme serves for as many values of t as is needed. Second, the accuracy of a homogeneous scheme can be estimated independently of t, as is stated just below.

CLAIM 1.1: *If* $(U, \pi, Z(t \mid u))$ *is homogeneous and* $Z(t \mid u)$ *is nonnegative and continuous in* t, *then* $\delta_Z^2(w, t)$ *is bounded for all* $0 \le t < \infty$, *i.e., for any given capacities* w, *there exists a finite*

$$S_Z(n, w) = \sup\{\delta_N^2(w, t) : 0 \le t < \infty\}$$

independent of the estimated value Z *(so that, in particular, the "problem of small probabilities" is avoided).*

PROOF: Proof is straightforward. ■

Claim 1.1 assures, through the Tshebyshev inequality, that for any given n and capacities $w(e)$, and for any positive ε, δ, there exists a sample size N such that for all $0 \le t < \infty$, we have

$$1 - \varepsilon < \frac{\hat{Z}_N(t)}{Z(w, t)} < 1 + \varepsilon \tag{1.11}$$

with probability at least $1 - \delta$. As is well known, this is not the case for CMC. On the other hand, the problem of small probabilities can also be settled by nonhomogeneous schemes [5]. Two homogeneous schemes are suggested by Elperin, Gertsbakh, and Lomonosov [4].

In the present paper the conventional problem of estimating static network reliability $R(G, q)$ is extended to include other equilibrium parameters of renewable networks, such as mean durations T_{UP}, T_{DN} of the network up and down periods. We show that they can be expressed through $R(G, q)$ and a certain directional derivative of $R(G, q)$ with respect to q and suggest a homogeneous MC scheme for estimating this derivative.

In Section 4, behavior of the s.r.e. of the suggested MC scheme is briefly discussed. We show that for the overall connectedness criterion the s.r.e. of the nonreliability estimate is bounded by a polynomial of n, uniformly for all w. This result seems to be nontrivial for reliable networks (R close to 1). However, whether or not the suggested MC scheme has the same property for the connectedness probability remains unclear.

2. DIFFERENTIATION FORMULA FOR *R*

As in Elperin et al. [4], a partition $\sigma = \{V_1, \ldots, V_k\}$ of the node-set V is called *regular* (with respect to a given G) if the induced subgraphs $G(V_i)$ are connected. The subsets V_i are referred to as components of σ, the number k of the components is denoted by $|\sigma|$. Consider the lattice $\mathbb{L} = \mathbb{L}(G)$ of regular partitions, with the natural ordering: $\sigma < \sigma'$ means that σ' can be obtained by merging components of σ. The height $h(\sigma)$ of a partition σ is defined by $|V| - |\sigma|$.

Let E_σ denote the set of external edges of σ, i.e., the edges between components of σ, and put $I_\sigma = E - E_\sigma$. I_σ is the (unique) maximal subset $F \subseteq E$ satisfying $\langle F \rangle = \sigma$; it is called the *closure* of any such F.

We retain the notation UP for the set of partitions $\langle F \rangle$, $F \in UP$, and similarly for DN. For $\sigma \in DN$, let J_σ denote the set of external edges e such that $\langle I_\sigma + e \rangle \in UP$. The partitions with nonempty J_σ form the "border zone" of the operational criterion, denoted by DN^*.

Adopt the parametrization of Eq. (1.8) of the q-vector, and let ∇ denote the gradient operator with respect to the capacities $w(e)$, $e \in E$. Furthermore, let $b = \{b(e) : e \in E\}$ be an arbitrary vector on E; for a subset $X \subseteq E$, put $b(X) = \Sigma\{b(e) : e \in X\}$. Then

$$b \cdot \nabla R(G, q) = \sum_{\sigma \in DN^*} P_\sigma(G, q) b(J_\sigma). \tag{2.1}$$

Remark: Differentiation formulae of this type appear from time to time in the literature (see, e.g., Lomonosov [8]).

Proof of Eq. (2.1): Consider two q-vectors on E, say $q = \{q(e) : e \in E\}$ and $q' = \{q'(e) : e \in E\}$, and let qq' denote the vector of products $q(e)q'(e)$, $e \in$

E. In static random graphs, an edge with failure probability qq' is equivalent to two parallel edges, one with failure probability q and one with q'. Based on this remark, calculate $R(G, qq')$ by applying the total probability formula to the double-edged G, assuming that the q-copies of the edges are tried first, and afterward the q'-copies of the edges that failed in the first try. We have

$$R(G, qq') = \sum_{F \subseteq E} P(F, q) R(G/F, q'), \tag{2.2}$$

where G/F denotes the results of contracting F and deleting loops; clearly, G/F depends only on $\langle F \rangle$. Note that the operational criterion should be reformulated for G/F as the meet of UP and the upper cone of $\langle F \rangle$ in $\mathbb{L}(G)$. Now, Eq. (2.2) can be used in two ways for differentiating R leading to two different formulae. For our needs, let q' tend to unity along the given vector $\boldsymbol{b}$; namely, put $q'(e) = \exp[-b(e)t]$, so that $p'(e) = 1 - q'(e) = t \cdot b(e) + o(t)$.

For $F \subseteq E$, let $d(F)$ denote the distance (minimal number of moves) in $\mathbb{L}(G)$ from $\langle F \rangle$ to UP. Then, by Eq. (1.2), $R(G, F, q') = A(F) \cdot t^{d(F)}(1 + o(1))$, where

$$A(F) = \Sigma \left\{ \prod_{e \in X} b(e) : X \subseteq E(G/F), |X| = d(F), \langle F \cup X \rangle \in UP \right\}.$$

Clearly, both $d(F)$ and $A(F)$ depend only on $\langle F \rangle$. In these terms, UP and DN^* are, in respect, the sets of partitions $\langle F \rangle$ with $d(F) = 0$ and 1. Taking all this into account, we obtain

$$R(G, q \cdot \exp[-b \cdot t]) = R(G, q) + t \cdot \sum_{\sigma \in DN^*} P_\sigma(G, q) b(J_\sigma) + o(t),$$

and Eq. (2.1) is established. ∎

Comparing Eqs. (2.1) and (1.5), one finds that

$$\Phi(G, q) = \mu \cdot \nabla R(G, q), \tag{2.3}$$

where $\mu = \{\mu(e) : e \in E\}$.

3. MERGING PROCESS MONTE CARLO SCHEME FOR ∇R

The main technical difficulty in MC estimation of ∇R is hitting the border states of the operational criterion. We do this by using trajectories on the lattice $\mathbb{L}(G)$ (see Section 2). Let us write $\sigma \lhd \sigma'$ if $\sigma < \sigma'$ and there is no τ such that $\sigma < \tau < \sigma'$, so that $|\sigma'| = |\sigma| + 1$, and σ' is the result of merging exactly two components of σ. A *trajectory* in $\mathbb{L}(G)$ is a sequence $u = (\sigma_0, \sigma_1, \ldots, \sigma_k)$, where σ_0 is the bottom element of $\mathbb{L}(G)$ (the partition into singletons) and $\sigma_0 \lhd \sigma_1 \lhd \cdots \lhd \sigma_k$. Let U_σ denote the set of trajectories terminating in σ and U^* the set of "full" trajectories, such that $\sigma_0, \ldots, \sigma_{k-1} \in DN$ and $\sigma_k \in UP$.

Adopt a parametrization of the form of Eq. (1.8), and put $w(\sigma) =$

$\Sigma\{w(e): e \in E_\sigma\}$. Given a trajectory $u = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ put $w_i(u) = w(\sigma_i)$, and define probability of u by

$$\pi(u) = \prod_{i=1}^{k} \frac{w_{i-1}(u) - w_i(u)}{w_{i-1}(u)}. \tag{3.1}$$

Let us now interpret t as time, and suppose that at $t = 0$ there are no edges, and each edge e of G appears at a random moment $t(e)$ to stay forever. Assume that $t(e)$ is distributed as $1 - \exp[-w(e)t]$, and that these moments for all edges are independent. Put $E(t) = \{e \in E: t(e) < t\}$. Then the probability $P_\sigma(w, t)$ of partition σ in the static model equals the probability of $\langle E(t)\rangle = \sigma$ (for details, see Elperin et al. [4] and Lomonosov [8]). This approach enables the calculation of the probabilities $P_\sigma(w, t)$ by solving the Kolmogorov differential equations for the Markov process $\langle E(t)\rangle$. In this way one obtains, for a nonabsorbing state σ,

$$P_\sigma(w, t) = \sum_{u \in U_\sigma} \pi(u) P_\sigma(t \mid u), \tag{3.2}$$

where

$$P_\sigma(t \mid u) = \exp[-w(\sigma)\cdot t]^* \operatorname*{conv}_{0 \le i \le h(\sigma)-1} (1 - \exp[-w_i(u)t])$$

and, for the absorbing state UP,

$$R(w, t) = \sum_{u \in U^*} \pi(u) R(t \mid u), \tag{3.3}$$

where

$$R(t \mid u) = \operatorname*{conv}_{0 \le i \le |u|-1} (1 - \exp[-w_i(u)t]).$$

(Here * and conv mean convolution: For $f(t)$ and $g(t)$, $t \ge 0$, their convolution is defined by $(f^*g)(t) = \int_0^t f(t-s)\, dg(s)$.) Substituting Eq. (3.2) into Eq. (2.1) provides

$$\boldsymbol{b}\cdot\nabla R(w, t) = \sum_{\sigma \in DN^*} b(\sigma) \sum_{u \in U_\sigma} \pi(u) P_\sigma(t \mid u) = \sum_{u \in U^{**}} \pi(u) b(\bar\sigma_u) P_{\bar\sigma_u}(t \mid u), \tag{3.4}$$

where U^{**} denotes the set of trajectories terminating in DN^*, and $\bar\sigma_u$ means the last member of trajectory u. The sum of probabilities $\pi(u)$ in the latter expression is, in general, not 1 (but is never less).

To have a presentation in the form of Eq. (1.9), consider the full trajectories (see Section 2). Any trajectory u terminating in a nonabsorbing state σ can be identified with the bundle $B(u)$ of full trajectories whose starting segment of length $|u|$ coincides with u. Moreover, $\pi(u) = \Sigma\{\pi(v): v \in B(u)\}$. In terms of full trajectories, Eq. (3.4) takes the form

$$b \cdot \nabla R(w,t) = \sum_{u \in U^{**}} \left(\sum_{v \in B(u)} \pi(v) \right) b(\bar{\sigma}_u) P_{\bar{\sigma}_u}(t \mid u)$$

$$= \sum_{v \in U^*} \pi(v) \cdot \sum_{i=l(v)}^{|v|-1} b(\sigma_i(v)) \cdot P_{\sigma_i(v)}(t \mid v), \tag{3.5}$$

where for a full trajectory $v = (\sigma_0, \sigma_1, \ldots, \sigma_{k-1}, \sigma_k)$ we set $|v| = k$ and $l(v) = \min\{i : \sigma_i \in DN^*\}$. We have now $\Sigma\{\pi(v) : v \in U^*\} = 1$, so that Eq. (3.5) has the form of Eq. (1.9).

Thus, we have a representation

$$b \cdot \nabla R(w,t) = \sum_{u \in U^*} \pi(u) \cdot Z(t \mid u), \tag{3.6}$$

where

$$Z(t \mid u) = \sum_{i=l(u)}^{|u|-1} b(\sigma_i(u)) \cdot P_{\sigma_i(u)}(t \mid u). \tag{3.7}$$

In Elperin et al. [4], an MC scheme based on generating trajectories on $\mathbb{L}(G)$ with probabilities of Eq. (3.1) was introduced. It is called a merging process (MP) MC. In this work, we use the scheme of Elperin et al. [4] with trajectory functionals of the form of Eq. (3.7).

4. REMARKS ON EFFICIENCY OF THE MERGING PROCESS MONTE CARLO

In general, the supremum $S(n, w)$ of the s.r.e. is unbounded as a function of w, as the following example shows.

Example 4.1: $V = \{s,t,x\}, w(s,x) = w(t,x) = \varepsilon$, $w(s,t) = m - 2\varepsilon$; the operational criterion is s,t-connectedness so that UP consists of the partitions $\{V\}$ and $\langle(s,t)\rangle$. There are three trajectories:

$$u_1 = (\sigma_0, \langle(s,x)\rangle, \{V\}),$$

$$u_2 = (\sigma_0, \langle(t,x)\rangle, \{V\}), \text{ with } \pi(u_1) = \pi(u_2) = \varepsilon/m, \text{ and}$$

$$u_3 = (\sigma_0, \langle(s,t)\rangle), \text{ with } \pi(u_3) = 1 - 2\varepsilon/m.$$

Suppose, the MP MC is used for estimating the unreliability $Q(w,t)$. We have

$$Q(t \mid u_1) = Q(t \mid u_2) = 1 - (1 - \exp[-mt])^*(1 - \exp[-(m-\varepsilon)t])$$

$$= m\left\{\exp[-(m-\varepsilon)t]\, \frac{1 - e^{-\varepsilon t}}{\varepsilon} + \exp(-mt)\right\},$$

$$Q(t/u_3) = \exp[-mt], \quad \text{and}$$

$$Q(w,t) = \exp[-(m-\varepsilon)t] \cdot (2 - \exp[-\varepsilon, t]).$$

It is now easy to see that

$$\lim \delta_Q^2(w,t) = m/\varepsilon$$

is unbounded. On the other hand, for $t \to \infty$ and $\varepsilon \cdot t = \text{const}$, one has

$$\delta_Q^2(w,t) = O(t) = O(|\ln Q|).$$

The preceding example inspires the following question: Does the bound

$$\delta_Q^2(w,t) \le C|\ln Q|$$

hold for the MP MC estimate of the network down probability for an arbitrary criterion?

Remark 4.2: In light of Example 4.1, Claim 6.2 of Elperin et al. [4] should be corrected as follows: For a given n and a given operational criterion, the coefficient of variation $\delta_{\mathrm{MP}}^2(Q)$ is bounded uniformly for all $t \in [0,\infty)$ and all λ-vectors satisfying $\max \lambda(\cdot)/\min \lambda(\cdot) \le C$ (for any C).

The proof of Claim 6.2 given in Elperin et al. [4] fits the preceding formulation.

An operational criterion exists for which MP MC has an s.r.e. of Q uniformly bounded by a polynomial of n. This is the overall connectedness, or, slightly more general, being partitioned into at most k components (for any given k). This means that for such criterion the sample size N guaranteeing the relation (1.11) is at most a polynomial of n.

CLAIM 4.3: *The mean s.r.e. of* MP MC *for the network nonconnectedness probability is less than* n^2.

PROOF: We have

$$\delta_Q^2 = \frac{\Sigma_u \pi(u) \cdot Q(t\,|\,u)^2}{Q(w,t)^2} - 1 \le \frac{\text{upper bound on } Q(t\,|\,u)}{\text{lower bound on } Q(w,t)}. \tag{4.3.1}$$

Let r denote the minimum cut capacity in terms of the w-function, i.e.,

$$r = \min\{w(X,\bar{X}) : \varnothing \subset X \subset V\},$$

where $w(X,X) = \Sigma\{w(x,y) : x \in X, y \in X\}$, and put $x = \exp[-rt/2]$. Then (see Lomonosov [8]),

$$Q(t\,|\,u) \le 1 - (1-x)^n - nx(1-x)^{n-1}. \tag{4.3.2}$$

On the other hand, we have, trivially,

$$Q(w,t) \ge x^2. \tag{4.3.3}$$

Take Eqs. (4.3.2) and (4.3.3) for the upper and lower bounds in Eq. (4.3.1). We have then

$$\delta_Q^2 < \frac{1}{x^2}\,[1-(1-x)^n - nx(1-x)^{n-1}]$$

$$= \frac{1}{x^2}\sum_{k=2}^{n}\binom{n}{k}x^k\cdot(1-x)^{n-k}$$

$$= \sum_{j=0}^{n-2}\frac{n(n-1)}{(j+2)(j+1)}\binom{n-2}{j}x^j\cdot(1-x)^{n-2-j} \le \binom{n}{2},$$

and the assertion follows. ■

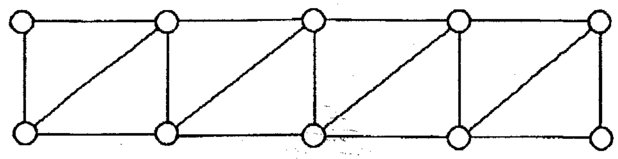
5. NUMERICAL EXAMPLES

We demonstrate in this section computations of reliability parameters for two networks with renewable edges. For these examples, we choose a ten-node series-parallel network ("ladder") used by Colbourn and Harms [3] for demonstrating the best up-to-date upper and lower bounds on all-terminal reliability. Our aim is to compare our MC estimates with those bounds (see Table 1). It was assumed that all edges have equal up probabilities $p(e) = p$ (column 1). The repair rates for all edges were assumed to be $\mu(e) = 1$, and the failure rates $\lambda(e)$ were determined from the equality $p(e) = 1/(1+\lambda(e))$. Two reliability parameters were computed by our MC method: the overall connectivity $R(G,q)$ and the stationary transition rate $\Phi = (T_{UP} + T_{DN})^{-1}$ (see Eq. (1.6)). Columns 2 and 3 show the best lower and upper bounds on $R(G,q)$ developed by Colbourn and Harms [3], and column 4 shows the actual value of $R(G,q)$. Column 5 presents our MC estimate obtained by averaging N independent replications (N shown in column 6). The computation of $R(G,q)$ was made using the representation of Eq. (3.3) and the edge evolution scheme with closure, the details of which are described by Elperin et al. [4]. It is worth noting that, in most cases, a relatively small number of replications N was needed to obtain an estimate of $R(G,q)$, which lies between fairly tight bounds *LB* and *UB*. Column 7 presents the estimates of the stationary transition rate $\Phi = (T_{UP} + T_{DN})^{-1}$, which was computed via the representation of Eq. (3.5) and by using the same set of trajectories as for $R(G,q)$. The relative error $\delta_{\hat{\Phi}}$ for $\hat{\Phi}$ is within 2–8% (column 8). The last two columns, 9 and 10, give the estimates of the mean equilibrium up and down times. They were computed by combining the formula for Φ and the expression $R(G,q) = T_{UP}/(T_{UP} + T_{DN})$.

As the network becomes more reliable ($p \to 1$), T_{DN} approaches 0.5. This can be explained by the fact that for R near 1, the dominant network failures are "cutting off" the upper-left or the lower-right nodes, and thus the corresponding repair rate is $\cong 2\mu(e) = 2$.

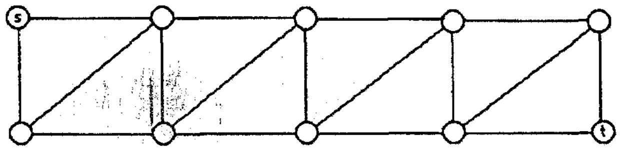
Table 2 presents our second example, which is the ten-node ladder now investigated for the s-t connectivity (see the figure in the upper part of Table 2). It was assumed $\lambda(e) = 10$ for vertical edges and $\lambda(e) = 20$ for diagonal edges.

TABLE 1. Reliability Parameters for All-Terminal Connectivity of a Ten-Node Ladder



$p(e)$	Best LB	Best UB	Actual $R(G,q)$	$\hat{R}(G,q)$	N	$\hat{\Phi} = [\hat{T}_{UP} + \hat{T}_{DN}]^{-1}$	$\delta_{\hat{\Phi}}\%$	$\hat{T}_{UP}$	$\hat{T}_{DN}$
1	2	3	4	5	6	7	8	9	10
0.1	$1.289\cdot10^{-6}$	$1.529\cdot10^{-6}$	$1.432\cdot10^{-6}$	$1.4\cdot10^{-6}$	1000	$1.084\cdot10^{-3}$	2	0.0132	9228
0.2	$3.140\cdot10^{-4}$	$4.416\cdot10^{-4}$	$3.906\cdot10^{-4}$	$3.4\cdot10^{-4}$	100	$1.047\cdot10^{-2}$	5	0.032	95.5
0.3	$5.502\cdot10^{-3}$	$9.049\cdot10^{-3}$	$7.651\cdot10^{-3}$	$6.77\cdot10^{-3}$	100	0.110	4	0.061	9.013
0.4	$3.233\cdot10^{-2}$	$5.999\cdot10^{-2}$	$4.948\cdot10^{-2}$	$4.482\cdot10^{-2}$	100	0.410	3	0.109	2.327
0.5	$1.052\cdot10^{-1}$	$2.049\cdot10^{-1}$	$1.700\cdot10^{-1}$	$1.578\cdot10^{-1}$	100	0.808	2.5	0.195	1.042
0.6	$2.423\cdot10^{-1}$	$4.415\cdot10^{-1}$	$3.841\cdot10^{-1}$	$3.648\cdot10^{-1}$	100	0.977	1.3	0.373	0.649
0.7	$4.500\cdot10^{-1}$	$7.081\cdot10^{-1}$	$6.409\cdot10^{-1}$	$6.216\cdot10^{-1}$	100	0.760	0.6	0.818	0.498
0.8	$7.081\cdot10^{-1}$	$8.712\cdot10^{-1}$	$8.548\cdot10^{-1}$	$8.426\cdot10^{-1}$	100	0.356	2.5	2.368	0.443
0.9	$9.302\cdot10^{-1}$	$9.733\cdot10^{-1}$	$9.710\cdot10^{-1}$	$9.675\cdot10^{-1}$	100	0.0732	6	13.22	0.444
0.94	$9.791\cdot10^{-1}$	$9.913\cdot10^{-1}$	$9.908\cdot10^{-1}$	$9.895\cdot10^{-1}$	100	0.0228	8.3	43.33	0.460
0.96	$9.922\cdot10^{-1}$	$9.964\cdot10^{-1}$	$9.962\cdot10^{-1}$	$9.962\cdot10^{-1}$	1000	0.00821	3.3	121.4	0.465
0.98	$9.9855\cdot10^{-1}$	$9.9914\cdot10^{-1}$	$9.9913\cdot10^{-1}$	$9.9912\cdot10^{-1}$	1000	0.00184	4	543.3	0.481
0.99	$9.99714\cdot10^{-1}$	$9.99793\cdot10^{-1}$	$9.99791\cdot10^{-1}$	$9.99788\cdot10^{-1}$	1000	0.000433	4	2308	0.490

TABLE 2. Reliability Parameters for a *s-t* Connectivity in a Ten-Node Ladder



$\mu(e) = 100$, $\lambda(e) = 10$ (vertical edges), $\lambda(e) = 20$ (diagonal edges)
$N = 10{,}000$ replications

$\lambda(e)$ for Horizontal Edges	$\hat{Q} = 1 - \hat{R}(G,q)$	$\delta_{\hat{Q}}\%$	$\hat{\Phi} = [\hat{T}_{UP} + \hat{T}_{DN}]^{-1}$	$\delta_{\hat{\Phi}}\%$	$\hat{T}_{UP}$	$\hat{T}_{DN}$
1	2	3	4	5	6	7
5	0.01078	1.1	2.070	1.1	0.478	0.00521
10	0.02427	1.0	4.818	0.9	0.203	0.00504
20	0.05458	0.7	11.41	0.7	0.0829	0.00478
100	0.69608	0.7	106.9	0.2	0.00284	0.00651
200	0.69794	0.8	106.8	0.2	0.00283	0.00654
400	0.69937	0.7	107.4	0.2	0.00280	0.00651
600	0.69974	0.7	107.4	0.2	0.00280	0.00651
1000	0.70016	0.7	107.3	0.2	0.00279	0.00652

The failure rate for horizontal edges changed from 5 to 1000 (see column 1). The edge renewal rate was taken $\mu(e) = 100$ for all edges. The estimates of network failure probability (loss of *s-t* connectivity) is given in column 2. It ranges from 0.00108 to 0.7 as the horizontal channels become less available. The estimate of the stationary transition rate $\hat{\Phi}$ is given in Column 4. The corresponding relative errors for $N = 10{,}000$ replications are quite small and are below 1% (see columns 3 and 5). The estimates of the mean up and down times are in columns 6 and 7. As $\lambda(e)$ grows, the prevalent connection mode between *s* and *t* becomes a path without horizontal edges. Thus, with a large probability, the *s-t* connection is restored by renewing a single diagonal or vertical edge. This explains the fact that T_{DN} lies below $[\mu(e)]^{-1} = 0.01$.

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