

Track Length Estimation Applied to Point Detectors

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Abstract—The concept of the track length estimator is applied to the uncollided point flux estimator (UCF) leading to a new algorithm of calculating fluxes at a point. It consists essentially of a line integral of the UCF, and although its variance is unbounded, the convergence rate is that of a bounded variance estimator. In certain applications, involving detector points in the vicinity of collimated beam sources, it has a lower variance than the "once-more-collided" point flux estimator, and its application is more straightforward.

Application de l'estimation de la longueur de trajet à un détecteur ponctuel

Résumé—On applique le concept de l'estimateur de longueur de trajet à l'estimateur du flux ponctuel n'ayant souffert aucune collision (UCF), et on aboutit à un algorithme nouveau pour le calcul des flux en un point. Cela consiste essentiellement en une intégrale curviligne de l'UCF, et bien que sa variance soit non bornée, le taux de convergence est celui d'un estimateur à variance bornée. Dans certaines applications utilisant des détecteurs ponctuels au voisinage de sources collimatées, il a une variance moindre que celle d'un estimateur du flux ponctuel ayant souffert une collision, et son usage est plus direct.

Pfad-Längen-Schätzung angewandt an punktförmigen Detektoren

Zusammenfassung—Das Konzept eines Pfad-Längen Schätzers ist auf den "uncollided flux" (UCF) Schätzer angewandt, was zu einem neuen Algorithmus zur Berechnung des Flusses an einen Punkt führt. Es besteht im wesentlichen aus einem Linienintegral des UCF. Obwohl die Varianz des Schätzers unbeschränkt ist, ist die Konvergenzrate die eines Schätzers mit beschränkter Varianz. In Anwendungen, die Detektorpunkte in der Nähe von kollimierten Strahlen als Quelle betrachten, hat das Verfahren eine geringere Varianz als der Ein-Stoß Punkt-Fluß Schätzer; es ist darüber hinaus in der Anwendung einfacher und direkter.

I. INTRODUCTION

The problem of Monte Carlo estimation of the flux at a point has been given considerable attention in the past. A number of methods for making the

estimates have been proposed,¹⁻⁶ most of which are of a specialized nature or difficult to implement in a multipurpose general code. The most widely used estimators are still the uncollided flux estimator (UCF) and the once-collided flux estimator (OCF) of Ref. 1.

The UCF involves the scoring of the expected uncollided contribution from every collision point to the detector point. It is an attractive estimator, because it is very simple to use and implement and requires relatively little data-handling and additional computation time. Its main and sometimes crucial disadvantage is its unbounded variance. The estimator has a singularity of the form $1/r^2$ (r being the distance from the collision point to the detector). It has been shown^{1,7} that, due to this singularity, the probability density function (pdf) of the sample average of this estimator does not follow the central limit theorem and, hence, does not converge to a normal distribution. Instead, as the sample increases, it converges to a pdf of the form

$$\phi^{\text{UCF}}(x) = \frac{1}{\pi} \int_0^\infty \exp(-\omega^{3/2}) \cos(x \cdot \omega + \omega^{3/2}) d\omega . \quad (1)$$

This pdf is asymmetrical around the origin and with $F(x)$ denoting its cumulative distribution,

$$F(x) = \int_{-\infty}^x \phi^{\text{UCF}}(t) dt ,$$

we obtain $F(0)/F(\infty) \cong \frac{2}{3}$. This also explains the often observed fact that the UCF tends to underestimate the flux. If we denote the sample average of the UCF by $\langle \xi^{\text{UCF}} \rangle$, then a confidence interval can be obtained from the relation⁷

$$\text{Prob} \left[\left| \frac{\langle \xi^{\text{UCF}} \rangle - I}{I} \right| \leq \frac{x}{N^{1/3}} \right] \cong F(x) - F(-x) , \quad (2)$$

where

I = exact flux at the point

N = number of histories used in the calculation

$x/N^{1/3}$ = confidence interval where x is chosen so as to reach a desired confidence level.

The value $F(x)$ is tabulated in Ref. 7.

Equation 2 indicates that the confidence interval converges as $N^{-1/3}$ rather than as $N^{-1/2}$, which is the case for a bounded variance estimator. This slow rate of convergence renders the estimator highly unreliable in those cases where the region close to the detector contributes a nonnegligible part of the flux.

The OCF estimator scores at every collision point a statistical one-sample estimate of the expected once-collided contribution of that point to the detector.^a This estimator has a bounded variance and thus converges as $N^{-1/2}$, which presents a major advantage

over the UCF. It involves, however, more complicated sampling and more cross-section data handling, which considerably increases the calculation time. This is more of a nuisance when many collisions occur far from the detector, where the UCF would be fast and reliable. Ideally one would prefer to use the OCF in the region enclosing the detector, and the UCF elsewhere. Such a simple combination of the two estimators is unfortunately impossible, because the once-collided contribution from a region involves intermediate collision points that may occur outside that region, which means that part of the uncollided contribution of the external region will be counted twice, leading to a systematic error.

The motivation for this work was the possibility of devising an estimator consisting of a line integral of the UCF in a limited region. This had two possible *a priori* advantages. First, a line integral of a $1/r^2$ behavior may reduce the singularity. Second, such an estimator can be combined with the UCF as described above. The natural approach to such an estimator would be the application of the track length estimation principle to the UCF estimator.

As is shown later in this paper, this indeed brings about a new estimator consisting basically of a line integral of the UCF. As expected, the radial singularity is reduced, but due to the fixed path direction, an angular singularity is introduced. However, the estimator does obey the central limit theorem and converges as $N^{-1/2}$. The formulation of the new estimator, statistical comparative analysis of its convergence characteristics, and the numerical results are presented in the following.

II. TRACK LENGTH ESTIMATION OF THE POINT FLUX

Let $g(x)$ denote the estimator or the response function, which will mean that at each collision point $x_i = (r_i, E_i, \Omega_i)$ in the region enclosing the detector, the quantity $g(x_i)$ is scored.

Consider two consecutive collision points in the detector region, x_i and x_{i+1} . In the present notation, E_i and Ω_i are the energy and direction with which the particle enters the collision at r_i , and the particle comes out of the collision with energy E_{i+1} and direction Ω_{i+1} , which are its energy and direction along the free flight path up to the next collision point $x_{i+1} = (r_{i+1}, E_{i+1}, \Omega_{i+1})$. The line of flight from r_i to r_{i+1} can be written as $r = r_i + t \cdot \Omega_{i+1}$, where t is a parameter, and if $d = |r_{i+1} - r_i|$, then $r_{i+1} = r_i + d \cdot \Omega_{i+1}$. Every phase space point along the path from x_i to x_{i+1} is thus given by $x = (r_i + t \cdot \Omega_{i+1}, E_{i+1}, \Omega_{i+1})$. The track length estimator of the detector response $g(x)$ will be given by the integral of $g(x)$ along the path from x_i to x_{i+1} in the form

^aA more detailed description of the OCF is given in Sec. IV.

$$G(d) = \int_0^d \Sigma_t(\mathbf{r}_i + t \cdot \boldsymbol{\Omega}_{i+1}) \times g(\mathbf{r}_i + t \cdot \boldsymbol{\Omega}_{i+1}, E_{i+1}, \boldsymbol{\Omega}_{i+1}) dt ; \quad (3)$$

both $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ may be points on the surface where the particle enters or leaves the detector region. If a particle has m straight paths in the detector region, then the quantity

$$\eta = \sum_{j=1}^m G(d_j) \quad (4)$$

is an unbiased estimator of the target response

$$I = \int_{\mathbf{x}} g(\mathbf{x}) \psi(\mathbf{x}) d\mathbf{x} , \quad (5)$$

where $\psi(\mathbf{x})$ is the collision density and the integration is carried over to the detector region. The fact that η of Eq. (4) is an unbiased estimator of I of Eq. (5) was proven in Ref. 8 for the case that $g(\mathbf{x})$ is constant along the flight path, and the general case was proven in Ref. 9. In Appendix A we present an alternate proof for the general case, which follows the method used in Ref. 8 for introducing a fictitious cross section and observing the limit as this cross section increases to infinity.

To apply the track length estimation to the UCF approach, we have to introduce into Eq. (3) the response function of the UCF and perform the integration. The detector response function in this case is given by

$$g(\mathbf{x}_i) = P(\boldsymbol{\Omega}_i \rightarrow \boldsymbol{\Omega}_d; E_i \rightarrow E_d) \frac{\Sigma_s(\mathbf{x}_i)}{\Sigma_t(\mathbf{x}_i)} \exp \left[- \int_0^{|r_d - r_i|} \Sigma_t(\mathbf{r}_i + \boldsymbol{\Omega}_d \cdot s) ds \right] \times \frac{1}{|\mathbf{r}_i - \mathbf{r}_d|^2} , \quad (6)$$

where

$\mathbf{x}_i = (\mathbf{r}_i, \boldsymbol{\Omega}_i, E_i)$ = phase space point at which the particle enters the collision

Σ_s/Σ_t = scattering probability at $(\mathbf{x}_i)$.

The value $P(\boldsymbol{\Omega}_i \rightarrow \boldsymbol{\Omega}_d; E_i \rightarrow E_d)$ is the pdf for scattering from the incoming direction $\boldsymbol{\Omega}_i$ into the direction $\boldsymbol{\Omega}_d$ leading toward the detector point and for the energy change from E_i to E_d , which is the energy with which the uncollided part will reach the detector; $\mathbf{r}_i$ and $\mathbf{r}_d$ are the spatial positions of the collision point and the detector, respectively.

Consider first the simplest case that can be treated analytically, that of a homogeneous medium with isotropic scattering in the laboratory system. In this case, all the cross sections are constant, the angular scattering density is simply $1/4\pi$ and $g(\mathbf{x})$ takes the form

$$g(\mathbf{x}_i) = \frac{\Sigma_s}{\Sigma_t} \frac{\exp[-\Sigma_t |\mathbf{r}_i - \mathbf{r}_d|]}{4\pi |\mathbf{r}_i - \mathbf{r}_d|^2} . \quad (7)$$

Introducing now the notation of Fig. 1, with r_1 being the distance between the first collision point and the detector point (r_d), ($r_1 = |\mathbf{r}_d - \mathbf{r}_i|$), and γ_1 the angle between $\mathbf{r}_d - \mathbf{r}_i$ and $\mathbf{r}_{i+1} - \mathbf{r}_i$, we obtain

$$\Sigma_t \cdot g(\mathbf{r}_i + t \cdot \boldsymbol{\Omega}_{i+1}) = \frac{\Sigma_s}{4\pi} \frac{\exp[-\Sigma_t \cdot l(t)]}{l^2(t)} , \quad (8)$$

where

$$l(t) = |\mathbf{r}_d - (\mathbf{r}_i + t \cdot \boldsymbol{\Omega}_{i+1})| = (r_1^2 + t^2 - 2r_1 \cdot t \cdot \cos \gamma_1)^{1/2} ,$$

and we thus have to calculate the integral

$$G(d) = \frac{\Sigma_s}{4\pi} \int_0^d \frac{\exp[-\Sigma_t \cdot l(t)]}{l^2(t)} dt . \quad (9)$$

The preceding integral also corresponds to the uncollided flux contribution of a line source to a point. An easy fast converging series can be obtained for this integral. Note that $G(d)$ can be separated into two parts, first integrating over t from 0 to $|r_1 \cdot \cos \gamma_1|$ (point a in Fig. 1) and then from $|r_1 \cdot \cos \gamma_1|$ to d . To account for all possible values of $\cos \gamma_1$ and $\cos \gamma_2$, since $\mathbf{x}_{i+1}$ may be to the left of point a when $\cos \gamma_2 < 0$, the division is done using the sign (x) function such that

$$G(d) = \left[\text{sign}(\cos \gamma_1) \int_0^{|r_1 \cos \gamma_1|} \frac{\exp[-\Sigma_t l_1(t)]}{l_1^2(t)} dt + \text{sign}(\cos \gamma_2) \int_0^{|r_2 \cos \gamma_2|} \frac{\exp[-\Sigma_t l_2(t)]}{l_2^2(t)} dt \right] \frac{\Sigma_s}{4\pi} \quad (10)$$

where

$$\text{sign}(\cos \gamma) = \begin{cases} -1 & \cos \gamma < 0 \\ 1 & \cos \gamma > 0 \end{cases} .$$

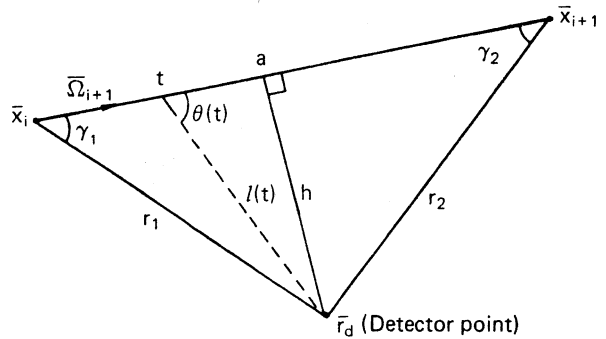


Fig. 1. Uncollided contribution to a detector point along the flight path between collision points.

Also,

$$l_1(t) = [r_1^2 + t^2 - 2r_1 \cdot t |\cos \gamma_1|]^{1/2}$$

and

$$l_2(t) = [r_2^2 + t^2 - 2r_2 \cdot t |\cos \gamma_2|]^{1/2}.$$

Using now the transformation $y = [r_1 |\cos \gamma_1| - t]$ in the first integral and $y = [r_2 |\cos \gamma_2| - t]$ in the second, we obtain

$$G(d) = \{ \text{sign}(\cos \gamma_1) B[|r_1 \cdot \cos \gamma_1|] + \text{sign}(\cos \gamma_2) B[|r_2 \cdot \cos \gamma_2|] \} \frac{\Sigma_s}{4\pi}. \quad (11)$$

With

$$B(L) = \int_0^L \frac{\exp[-\Sigma_t(y^2 + h^2)]}{y^2 + h^2} dy, \quad h = r_1 \sin \gamma_1,$$

expression (10) amounts simply to performing the integration from point a to r_i and r_{i+1} , respectively.

Using now the transformation $s^2 = y^2 + h^2$, $B(L)$ can be expanded as

$$\begin{aligned} B(L) &= \int_h^{(L^2+h^2)^{1/2}} \frac{\exp(-\Sigma_t \cdot s)}{s(s^2 - h^2)^{1/2}} ds \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \Sigma_t^n}{n!} \int_h^{(L^2+h^2)^{1/2}} \frac{s^{n-1}}{(s^2 - h^2)^{1/2}} ds \\ &= \sum_{n=0}^{\infty} B_n, \end{aligned} \quad (12)$$

where the last equality is obtained by expanding the exponent in a Taylor series. The first three terms of the sum are easily calculated by direct integration, yielding

$$B_0 = \left\{ \frac{\pi}{2} - \arcsin \left[\frac{h}{(L^2 + h^2)^{1/2}} \right] \right\} \frac{1}{h},$$

$$B_1 = -\Sigma_t \ln \left[\frac{L + (L^2 + h^2)^{1/2}}{h} \right],$$

and

$$B_2 = \frac{\Sigma_t^2}{2} \cdot L,$$

and higher order terms can be deduced from the recursion relation obtained through integration by parts

$$\begin{aligned} B_n &= (-1)^n \frac{\Sigma_t^n \cdot L}{(n-1)n!} (L^2 + h^2)^{(n-2)/2} \\ &+ \frac{(n-2)\Sigma_t^2 h^2}{(n-1)^2 n} B_{n-2}. \end{aligned} \quad (13)$$

The values $B[|r_1 \cdot \cos \gamma_1|]$ and $B[|r_2 \cdot \cos \gamma_2|]$ can now be easily calculated and used in Eq. (11) to obtain the flux estimate for this case.

In most practical cases, energy and angular dependence as well as heterogeneous media are involved, and analytic integration of Eq. (3) with $g(\mathbf{x})$ of Eq. (6) is impossible. In such cases we can use a one-sample Monte Carlo estimate of $G(d)$. This is similar to the approach used in the case of the OCF, where the integrated once-collided contribution of a collision point to the detector point is generally impossible to calculate even in the homogeneous case, and a one-sample estimate of that integral is used.¹ This can be done in the following manner. The integral of Eq. (3) is rewritten in the form

$$G(d) = \int_0^d \frac{\Sigma_t(r_i + t \cdot \Omega_{i+1}, E_{i+1}) g(r_i + t \cdot \Omega_{i+1}, E_{i+1}, \Omega_{i+1})}{q(t)} \times q(t) dt, \quad (14)$$

where $q(t)$ is any normalized pdf in the range $[0, d]$. To get a one-sample estimate of $G(d)$, a realization t_b of t is sampled from $q(t)$, and the score of the track from $\mathbf{x}_i$ to $\mathbf{x}_{i+1}$ is then taken as

$$\tilde{G}(d) = \frac{\Sigma_t(r_i + t_b \cdot \Omega_{i+1}, E_{i+1}) g(r_i + t_b \cdot \Omega_{i+1}, E_{i+1}, \Omega_{i+1})}{q(t_b)}. \quad (15)$$

To complete the procedure, we have to choose a suitable pdf, $q(t)$. It is desirable to choose $q(t)$ in a form that will reduce the singularity in $g(\mathbf{x})$. That singularity is of the form

$$|(\mathbf{r}_i + t_b \cdot \Omega_{i+1}) - \mathbf{r}_d|^{-2},$$

which from Fig. 1 is seen to be identical with $(r_1^2 + t_b^2 - 2r_1 \cdot t_b \cdot \cos \gamma_1)$. Thus, a natural choice of $q(t)$ will be

$$q(t) = \frac{c_1}{r_1^2 + t^2 - 2r_1 t \cos \gamma_1}, \quad 0 \leq t \leq d, \quad (16a)$$

with c_1 a normalization constant. The normalization of $q_1(t)$ in the range $[0, d]$ yields

$$\begin{aligned} c_1 &= r_1 \sin \gamma_1 \left[\arctg \left(\frac{d - r_1 \cos \gamma_1}{r_1 \sin \gamma_1} \right) \right. \\ &\quad \left. - \arctg(-\cotg \gamma_1) \right]^{-1}. \end{aligned} \quad (16b)$$

To sample t_b from $q_1(t)$, let ξ be a random number uniformly distributed in $[0, 1]$. Then from the equation

$$\int_0^{t_b} q(t) dt = \xi$$

we obtain

$$t_b = r_1 \cos \gamma_1 + r_1 \sin \gamma_1 \cdot \text{tang} \left[\frac{r_1 \sin \gamma_1 \cdot \xi}{c_1} + \text{arctg}(-\cot \gamma_1) \right]. \quad (17)$$

With t_b sampled from Eq. (17) and using the explicit form of $q(t)$, we obtain for the score

$$\tilde{G}(d) = \frac{\Sigma_t(\mathbf{r}_i + t_b \cdot \boldsymbol{\Omega}_{i+1}, E_{i+1}) g(\mathbf{r}_i + t_b \cdot \boldsymbol{\Omega}_{i+1}, E_{i+1}, \boldsymbol{\Omega}_{i+1}) (r_1^2 + t_b^2 - 2r_1 t_b \cos \gamma_1)}{c_1}. \quad (18)$$

It can be easily seen that the $|(\mathbf{r}_i + t_b \cdot \boldsymbol{\Omega}_{i+1}) - \mathbf{r}_d|^{-2}$ singularity that exists in $g(\mathbf{r}_i + t_b \cdot \boldsymbol{\Omega}_{i+1}, E_{i+1}, \boldsymbol{\Omega}_{i+1})$ is removed by the same factor in the numerator of $\tilde{G}(d)$. However, the division by c_1 introduces a singularity of the form $(r_1 \sin \gamma_1)^{-1}$. This singularity results in an unbounded variance, but as was shown in Ref. 7, although the variance is unbounded, the sample average of this estimator still has a pdf that converges to the normal distribution. If we denote the sample average of the track length point (TLP) estimator of Eq. (18) by $\langle \xi^{\text{TLP}} \rangle$, then it is shown in Ref. 7 that the confidence interval of that estimator can be determined from the relation

$$\text{Prob} \left[\left| \frac{\langle \xi^{\text{TLP}} \rangle - I}{I} \right| \leq \frac{x}{N^{1/2}} \right] \cong F^0(x) - F^0(-x), \quad (19)$$

where

$I = \text{exact result}$

$F^0(x) = \text{cumulative normal distribution.}$

Two points are worth noting in Eq. (19). Just as in Eq. (2), it does not contain the variance. This is to be expected since in both cases the variance is undefined. Unlike in Eq. (2), the confidence interval here converges as $N^{-1/2}$, which is the same rate of convergence as a bounded variance estimator.

We now review the considerations involved in implementing the TLP in a general purpose code. If the detector point is at $\mathbf{r}_d$, then we surround the detector point with a volume referred to as the detector volume, within which the TLP is activated. That is, when a collision occurs outside the detector volume, the UCF is used, and when any straight path of a particle flight occurs within that volume, the TLP is used. The volume can be any one of the zones naturally defined in the transport problem, or it can be a special zone defined for using the TLP. In most general purpose codes, each surface crossing made by a particle is identified, so that, if the particle is generated outside the detector volume, one does not have to worry about the TLP until the particle crosses the detector volume. Up to that point, the UCF is used, and since this involves only collision points that are outside the detector volume, the distance $|\mathbf{r}_d - \mathbf{r}_i|$ is always finite and the UCF has, in this case, a bounded variance and its sample average converges at a rate of $N^{-1/2}$.

When a particle crosses the detector volume, the beginning and the end of the straight path in the detector volume are recorded as $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ (this is in any case done by the code once more as a part of the particle's tracking). The quantities r_1 and γ_1 must be defined as

$$r_1 = |\mathbf{r}_d - \mathbf{r}_i|$$

and

$$\cos \gamma_1 = \frac{(\mathbf{r}_{i+1} - \mathbf{r}_i) \cdot (\mathbf{r}_d - \mathbf{r}_i)}{|\mathbf{r}_{i+1} - \mathbf{r}_i| \cdot |\mathbf{r}_d - \mathbf{r}_i|};$$

then c_1 is calculated [Eq. (16b)] and t_b is sampled by Eq. (17). The point $\mathbf{r}_b = \mathbf{r}_i + t_b \cdot \boldsymbol{\Omega}_{i+1}$ on the straight path is then defined and $\tilde{G}(d)$ is calculated from Eq. (15). Note that the major part of calculating $\tilde{G}(d)$ is in obtaining $g(\mathbf{r}_b, E_{i+1}, \boldsymbol{\Omega}_{i+1})$, which is just the application of the UCF at the phase space point $(\mathbf{r}_b, E_{i+1}, \boldsymbol{\Omega}_{i+1}) = \mathbf{x}_b$.

Thus we may conclude that the use of the TLP in combination with the UCF requires additional time only for sampling t_b and defining c_1 . No major geometry changes are required, and no additional cross-section data retrieval is necessary. Altogether we get an estimator that converges as $N^{-1/2}$.

An alternative, though not very different, sampling procedure for t_b can be used with

$$q_2(t) = \frac{c_2}{(r_1^2 + t^2 - 2r_1 t \cos \gamma_1)^{1/2}}, \quad 0 \leq t \leq d; \quad (20)$$

in this case, normalization yields

$$c_2 = \left\{ \ln \left[\frac{(r_1^2 + d^2 - 2r_1 d \cos \gamma_1)^{1/2} + (d - r_1 \cos \gamma_1)}{r_1(1 - \cos \gamma_1)} \right] \right\}^{-1} \quad (21)$$

and t_b is obtained from the relation

$$t_b = r_1 \cos \gamma_1 + \frac{(be^\alpha)^2 - a^2}{2be^\alpha};$$

$$b = r_1(1 - \cos \gamma_1),$$

$$a^2 = r_1^2 \sin^2 \gamma_1, \quad \alpha = \xi/c_2. \quad (22)$$

This procedure yields an estimator of the form

$$\tilde{G}(d) = \frac{\Sigma_t(\mathbf{x}_b) \cdot g(\mathbf{x}_b) (r_1^2 + t_b^2 - 2r_1 t_b \cos \gamma_1)^{1/2}}{c_2}. \quad (23)$$

This estimator involves the same type of singularity as that of Eq. (18), and in all the numerical studies carried out, the results of both procedures were similar.

It is interesting to note that whereas in the case of the OCF one could choose a sampling procedure for an intermediate collision point by reducing the r^{-2} singularity to r^{-1} , this seems to be impossible here, and an angular singularity ($r \sin \gamma_1$) is present. Presumably this is so because, unlike the case of the OCF, here the averaging of the pdf does not involve integration over the angles. It is averaged over a line with a fixed angle (γ_1). Thus the angular singularity cannot be removed.

Although it is shown in Ref. 7 by using classical limit theorems for the unbounded variance case that the convergence rate of the TLP is $N^{-1/2}$, we attempt to obtain a similar result in Sec. III by a statistical analysis based on the behavior of the characteristic function of the sample average of the TLP. Such analysis follows the same route as the analysis used in Ref. 1 to identify the convergence rate of the UCF.

III. STATISTICAL ANALYSIS OF THE TLP CONVERGENCE

The variance of both estimators [Eqs. (18) and (23)] behaves as the integral

$$\int_0^\pi \int_0^\infty \tilde{G}^2(d) r_1^2 dr_1 \sin \gamma_1 d\gamma_1 .$$

It can easily be seen that in both cases the radial part is bounded ($1/r$ behavior); however, the angular part yields an unbounded integral of the form

$$\int_0^\pi \frac{d\gamma}{\sin \gamma} .$$

This type of singularity seems to be inherent in the estimator, and no singular biasing can remove it since γ_1 is fixed for a given path. Thus, even though the type of singular biasing of $q(t)$ and $q_2(t)$ is very similar to the type used for the once-collided flux estimator, nevertheless, there the angle of the intermediate collision point is implicitly biased and its singularity removed. Here, the fixed nature of this angle does not allow such biasing. It is possible in principle to apply angular biasing at every collision point in a form similar to the angular resampling of Ref. 5. However, such biasing may require a considerable amount of additional calculation time.

A singularity of the type $1/\sin \gamma$ results when one calculates the average of a random variable proportional to x^{-1} with a distribution function proportional to x . We assume for convenience that x is distributed in the range $[0, 1]$ with pdf

$$p(x) = 2x ,$$

and we look at the average of the random variable $y(x)$,

$$y(x) = (2x)^{-1} ,$$

clearly then,

$$\langle y \rangle = \int_0^1 y(x) p(x) dx = 1$$

and

$$\langle y^2 \rangle = \int_0^1 y^2(x) p(x) dx = \int_0^1 \frac{dx}{2x} = \infty .$$

We are interested in the convergence properties of the mean of the variable $y(x)$. Its distribution function is given by

$$q(y) = \frac{1}{2} y^{-3} , \quad \frac{1}{2} \leq y < \infty ,$$

and in order to learn the behavior of the sum of many independently calculated values of y , it is necessary to know the characteristic function $\phi_y(t)$, where

$$\phi_y(t) = \frac{1}{2} \int_{1/2}^\infty \frac{\exp(iyt)}{y^3} dy = \frac{t^2}{2} \int_{1/2}^\infty \frac{\exp(iu)}{u^3} du .$$

The exponent is displayed as $(\cos u + i \sin u)$, and the integrals can be found in standard tables. These integrals involve the Si and Ci functions for which series expansions are available. One thus obtains

$$\begin{aligned} \phi_y(t) = & \left(\cos \frac{t}{2} - \frac{t}{2} \sin \frac{t}{2} + \frac{t^2}{4} \ln \left(\frac{\gamma t}{2} \right) - \frac{t^4}{64} + \dots \right. \\ & \left. + i \left(\sin \frac{t}{2} + \frac{t}{2} \cos \frac{t}{2} - \frac{\pi}{4} t^2 + \frac{t^3}{8} - b t^5 + \dots \right) ; \right. \end{aligned}$$

for small values of t ,

$$\begin{aligned} \phi_y(t) = & 1 + i \cdot t + \frac{t^2}{4} \ln t + \delta t^2 \\ & + \dots (\text{higher order terms } -t^n) \text{ and} \end{aligned}$$

$$\ln \phi_y(t) = i \cdot t + \frac{t^2}{4} \ln t + \delta t^2 + \dots .$$

If we make n independent observations of y and consider the centralized sample average

$$Y = \frac{\sum_{i=1}^n y_i}{n} - 1 ,$$

then the characteristic function of Y is given by

$$\phi_Y(t) = \exp(-it) [\phi_y(t/n)]^n ,$$

and thus,

$$\begin{aligned}\ln \phi_Y(t) &= -i \cdot t + n \ln[\phi_Y(t/n)] \\ &= -it + it + \frac{t^2}{4n} \ln(t) - \frac{t^2}{4n} \ln n + \frac{\delta t^2}{n} + \dots\end{aligned}$$

Thus $\ln \phi_Y(t)$ converges to zero as n increases, and the slowest converging term behaves as $\ln(n)/n$.

The variable Y has the form of $(y - \langle y \rangle)$, where $\langle y \rangle$ is the theoretical average and y is the sample average of the n observations. Note that in general, if one has a random variable $z(n)$ depending on an integral parameter n with a pdf $p[z(n)]$, then the characteristic functions will be a function of n . If, as n goes to infinity, the distribution $p[z(n)]$ narrows to a delta function, then the characteristic function tends to unity for every value of t (or its logarithm to zero) and vice versa. Thus the rate at which the logarithm of the characteristic function converges to zero as $n \rightarrow \infty$ can be taken as a measure of the decrease in the width of the distribution function of y . In the case of the UCF, it was shown¹ that the characteristic function of Y has the form

$$\ln \phi_Y^{\text{UCF}}(t) = \gamma^1 n^{-1/2} t^{3/2} + \delta^{-1} n^{-1} t^2$$

and thus converges to zero as $n^{-1/2}$. In our case, the convergence rate is $n^{-1} \ln(n)$, which is, by an infinitesimal factor, close to n^{-1} [i.e., $n^{-1} \ln(n)$ converges to zero faster than $n^{-\epsilon-1}$ for any positive infinitesimal ϵ]. We can conclude that the average estimate of the TLP has a faster narrowing distribution and thus a better convergence characteristic.

IV. NUMERICAL EXAMPLE

The TLP was implemented in the TIMOC code.¹⁰ A test calculation was done to compare the TLP with the UCF and to validate its expected convergence properties. The problem consists of an iron sphere with a 20-cm radius and a californium point source in the center. Fifty-six Eurlib cross-section groups were used in the calculation and the target response was the flux at a point on the surface of the sphere in the 0.674- to 14.92-MeV energy range. In this case, no combination of the TLP and UCF was used, and when the TLP was used, the whole sphere served as a detector volume. Let us first discuss the results obtained for the TLP. The results of Ref. 7 indicate that in a region where the angle γ_1 (see Fig. 1) is small, the TLP estimator converges to a normal variable. Since, outside that cone, the TLP has a bounded variance, the overall estimator can be viewed as a sum of the two normal variables, yielding a normal variable. Thus the error analysis applied to the TLP is the same as for any bounded variance estimator. In order to verify the normal behavior of the TLP, the results of the calculation were obtained in 40 batches, where each

batch was the sample average of contributions of 500 histories. The batch results are shown in Table I, where ξ_j^b is the sample average of the j 'th batch multiplied by 10^4 .

The batch results were tested for normality using the χ^2 test¹¹ and the Shapiro-Wilk W test.¹² The reduced χ^2 , with 38 degrees of freedom, is 0.373, indicating very good agreement with a normal distribution, and for the W test, the results yield $W = 0.9526$, again in good agreement with normal behavior. The estimate of the flux is given by

$$\langle \xi \rangle = \sum_{j=1}^{40} \xi_j^b / 40 = 2.416 \times 10^{-4} \text{ n/cm}^2 \cdot \text{s} ,$$

which is, of course, identical with the "single particle" estimated flux, and the estimate of the variance is

$$V_{\langle \xi \rangle} = \frac{1}{40} \cdot \frac{1}{39} \cdot \sum_{j=1}^{40} (\xi_j^b - \langle \xi \rangle)^2 = 7.179 \times 10^{-11} .$$

The percentage relative standard deviation (PRSD) is then given by

$$\text{PRSD} = 100 \frac{\sqrt{V_{\langle \xi \rangle}}}{\langle \xi \rangle} = 3.507\% .$$

The interpretation of the PRSD as a measure of the error is that, assuming that the batch variable is indeed

TABLE I
Track Length Point Estimators for ²⁵²Cf Neutrons
in a 40-cm Radius from Sphere

j	ξ_j^b ($\times 10^4$)	j	ξ_j^b ($\times 10^4$)
1	1.625	21	2.30
2	1.642	22	2.302
3	1.7	23	2.323
4	1.762	24	2.458
5	1.86	25	2.487
6	1.92	26	2.518
7	1.943	27	2.605
8	1.98	28	2.664
9	2.00	29	2.692
10	2.001	30	2.704
11	2.026	31	2.722
12	2.092	32	2.742
13	2.14	33	2.80
14	2.175	34	3.045
15	2.192	35	3.12
16	2.201	36	3.177
17	2.212	37	3.199
18	2.24	38	3.294
19	2.292	39	3.336
20	2.293	40	3.958

a normal variable, then with probability 0.682, the estimated result is within 3.5% of the exact result. The quality factor of every estimator (with a limiting normal behavior) is defined as $q = (\text{PRSD})^2 \cdot T$, where T is the overall time required for the calculation. In the present case, $T = 215$ s and thus $q^{\text{TLP}} = 2644$ (q can be interpreted as the calculation time required to obtain a 1% PRSD).

The situation for the UCF is considerably different, because the UCF does not converge to a normal variable. A batch analysis similar to the above for the UCF gave $\chi^2 = 1.84$ and $W = 0.853$, indicating an abnormal behavior. To obtain a confidence interval for the UCF, we used Eq. (2) and the tabulation of $F(x)$ (Ref. 7) to obtain a confidence interval of 0.68 probability (with the assumption that the pdf of a single-particle score is already the limiting distribution). This yields

$$\text{Prob} \left[\left| \frac{\langle \xi^{\text{UCF}} \rangle - I}{I} \right| \leq \frac{1.8}{\sqrt[3]{20\,000}} \right] \cong 0.68 ,$$

where I is the exact result and $\langle \xi^{\text{UCF}} \rangle$ is the calculated flux estimate, 2.216×10^{-4} n/cm²·s. The 0.68 confidence interval is thus $0.0663 \cdot I$, and the PRSD with the same interpretation as above will be

$$\text{PRSD} = 100 \cdot \frac{0.0663 \cdot I}{I} = 6.63\% .$$

In defining the quality factor for the UCF calculation, recall that it does not converge as $N^{-1/2}$ but as $N^{-1/3}$; thus, in order to get a quantity independent of the number of histories, we must define in this case

$$q_{\text{UCF}} = (\text{PRSD})^3 \cdot T_{\text{UCF}} ,$$

where q_{UCF} is again the time required to obtain a 1% confidence interval (for 0.68 confidence level). In the present case, the time required for running 20 000 histories was $T_{\text{UCF}} = 206$ s, and $q_{\text{UCF}} = 59\,224$ was obtained. The ratio $q^{\text{UCF}}/q^{\text{TLP}} = B$ may serve as a measure of the advantage of the TLP over the UCF. In this case, we obtain $B = 22.3$.

It is worth noting that the time required for the TLP calculation was a very small factor (1.043) larger than that for the UCF for the same number of source particles. This is in accordance with the previous discussion. The extra time for using the TLP, consisting of sampling t_b and calculating c_1 , is very small in relation to the time needed for the transport in a practical problem.

The next case was a deep penetration iron benchmark. A cross section of its geometry is shown in Fig. 2. In this experiment, the penetration of neutrons being emitted by a highly ²³⁵U-enriched uranium fission source into an iron shield was measured. Point fluxes were scored at eight detector positions situated

at various distances in the iron shield and in the disk source (see Fig. 2). In this calculation, all three methods, the UCF, OCF, and TLP, were employed. A total of 48 000 particles were used in each calculation, and the results were analyzed in batches of 2000 histories. We consistently found that for all the detectors, both the OCF and the TLP converge to a normal distribution, yet the OCF was always superior to the TLP, with its quality factor [$q = (\text{PRSD})^2 \cdot T$] being smaller than those of the TLP by factors ranging from 1.2 to 25. A typical example of the behavior of the cumulative flux estimate as a function of the number of histories is shown in Fig. 3. It is seen there that the OCF fluctuates considerably less than either the UCF or the TLP. It is worthwhile to mention that in these calculations the iron shield was subdivided into geometric zones, enclosing the detectors as shown in Fig. 2. In the TLP calculation, the UCF was used outside that zone and the TLP, inside. The original assumption that the improved convergence close to the detector combined with the reduced calculation time (relative to the OCF) would yield an overall improved quality was not fulfilled. In all cases, the time reduction was only marginal, and the better convergence characteristic of the OCF was the decisive factor. In this case, obviously, the time required for sampling an additional collision point is small relative to the other parts of the calculation. This situation, however, is very much dependent on the problem and the structure of the code in use. It should furthermore be emphasized that although the TLP does converge to a normal distribution, still not much can be said about the rate of that convergence or about the width of the distribution. Regarding the use of the TLP, it is also possible to consider various sizes of detector zones, i.e., changing the volume in which the TLP is scored. We have tried it in a number of cases involving cylindrical and spherical shielding geometries with isotropic point sources. The results indicated that the combination of the UCF and TLP is insensitive to an increase in the detector volume beyond a linear dimension of ~ 1 mfp. It seems that the process of finding an optimum detector volume may be time-consuming and offer only a marginal benefit.

Another numerical study was done on the case of a unidirectional source. An iron cylinder with a 5-cm radius and height $Z = 50$ cm was considered. A point source emitting fission neutrons in the z direction was positioned 1 mm off of the symmetry axis of the cylinder. The integrated flux in the 1- to 14.9-MeV range was calculated for seven detector points simultaneously in the same computer run. From each collision point, the score was calculated for all seven detectors. Problems involving a unidirectional source are frequently encountered in accelerator targets (like spallation sources) and in photon applications where objects are irradiated by a collimated beam. Such cases in general have an interesting feature with respect to

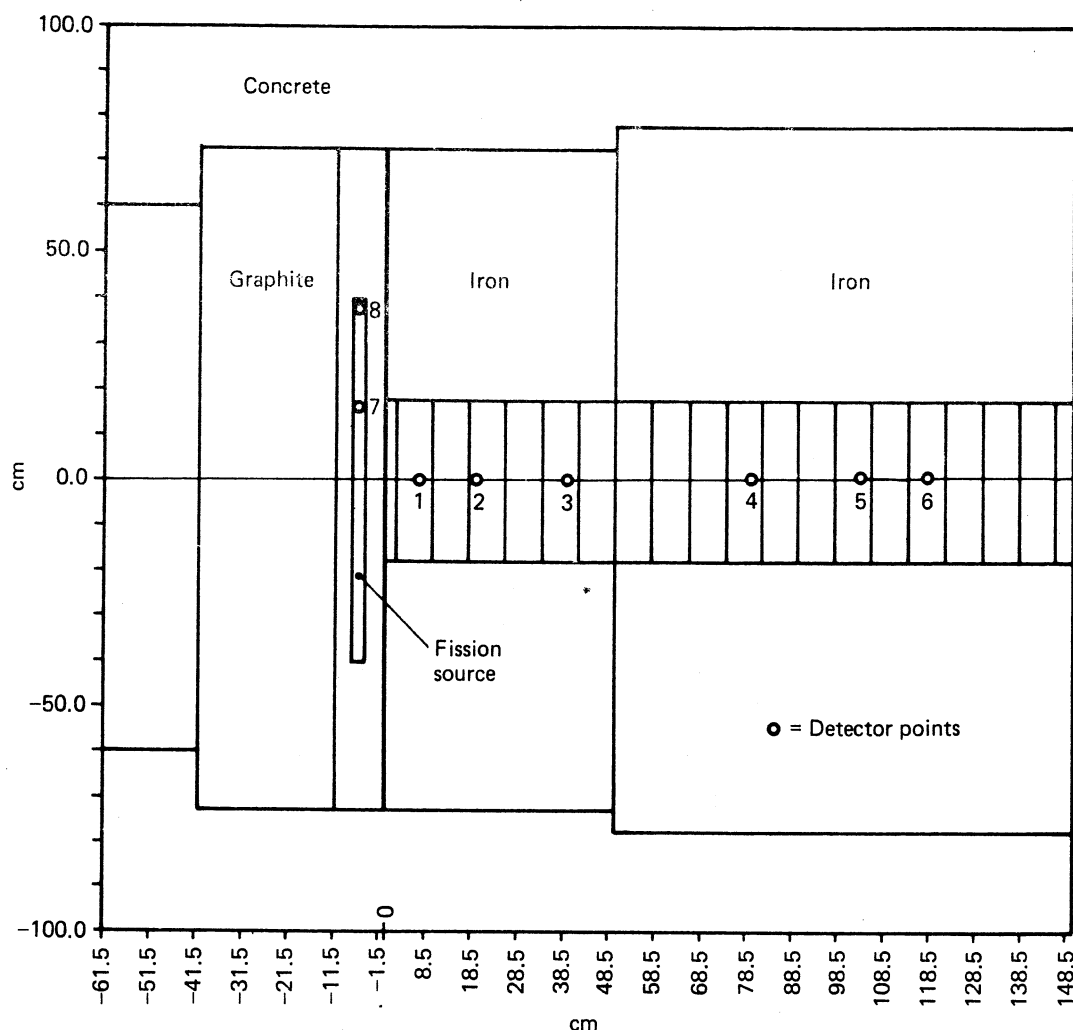


Fig. 2. Vertical cross section of the Monte Carlo mockup of the EURACOS iron benchmark experiment, indicating the point detector positions.

the effectiveness of the OCF estimator. In the application of the OCF at every collision point r_i , an intermediate collision point r_b is sampled from a singular geometrical pdf, and the expected once-collided flux of particles that will collide at r_b and then reach the detector point is scored. But it is also necessary to account for the direct uncollided contribution of the source to the detector point and for the once-collided contribution from the source, or else the first two Neumann series terms of the flux will not be accounted for. The direct uncollided contribution of the source must be scored by the UCF. In general shielding problems, this contribution is very small. However, in those cases where this contribution is large, the effectiveness of the OCF's reduced singularity is diminishing, because a major part of the scoring is actually done with the UCF.

The case of unidirectional source is of special importance in that respect, since the direct uncollided

source contribution to detector points, which are not on the line of the emission of the source, is zero and the once-collided source contribution *cannot* be sampled with the OCF. In such cases it is necessary to take the *uncollided* contribution of the first collision point, and in fact, the first collision density serves as the source. For detector points close to the emission line (or beam), the uncollided part (from the first collision source) will be large enough to reduce the effectiveness of the OCF. As a prototypical example, we studied the case of six detector points situated along the symmetry axis and one point off the axis. The results of 50 000 histories analyzed in batches of 2000 are shown in Table II. The results of the flux obtained by the UCF are also shown. The calculation time for the UCF (for 50 000 histories) was 5.31 min. The times required for TLP and OCF were 5.35 and 7.41 min, respectively. It again confirms the fact that the TLP and UCF require the same computing effort, whereas the OCF is

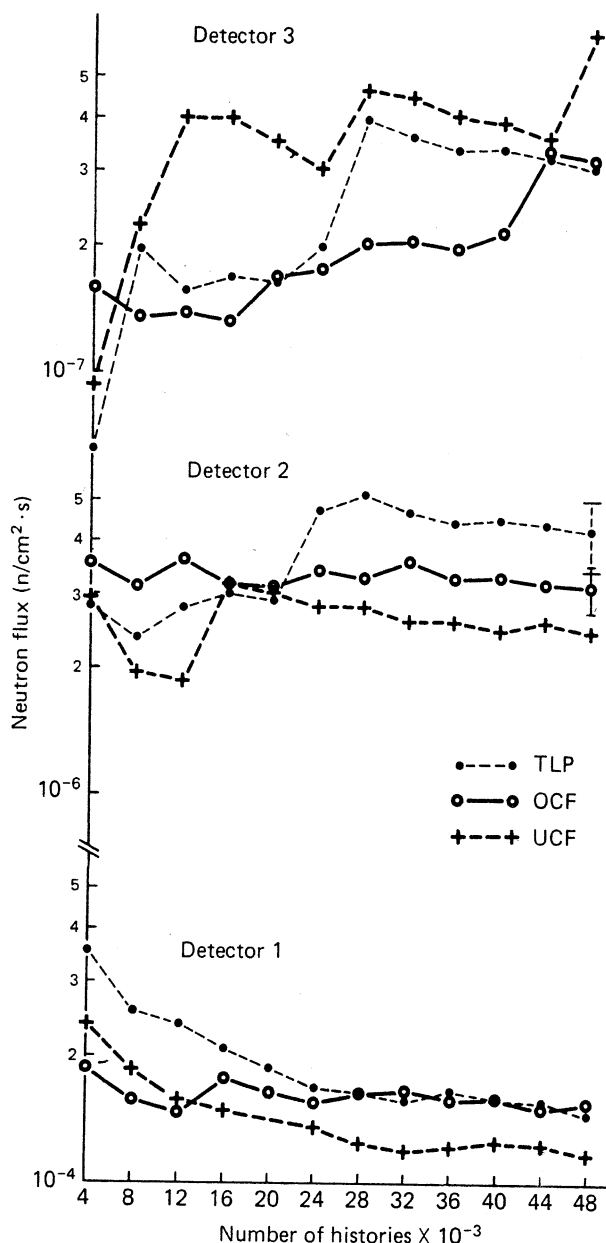


Fig. 3. Neutron flux estimates calculated in an iron benchmark experiment by three different estimators as a function of the number of histories.

more time-consuming. This fact is enhanced in the common case of many point detectors used in the same calculation because of an increased ratio of scoring time versus transport time. It is seen from Table II that the TLP is superior to the OCF by factors ranging from 1.17 to 7.9. It is interesting to note the detector point as (0, 4.9, 30) where the PRSD of the OCF is smaller than that of the TLP. This point is far enough from the first collision source so that the uncollided contributions do not affect the calculations severely. In Fig. 4, results are shown for the detector

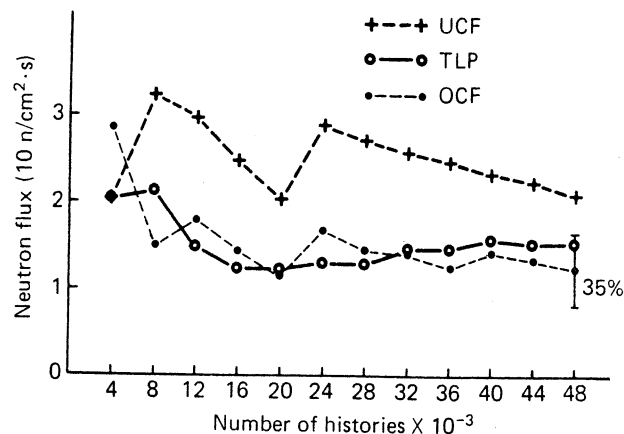


Fig. 4. Neutron flux estimates calculated at a point near the place where a collimated beam enters a medium.

at (0, 0, 30). Contrary to Fig. 3, one may note that the TLP is the least fluctuating, and again, the UCF is the most fluctuating.

V. CONCLUSIONS

A new point flux estimator was devised by applying track length estimation to the uncollided point estimator (UCF), resulting in an estimator that scores along the flight track of the particle. The new estimator has an interesting feature, being the only one known to us with an undefined variance and yet an $N^{-1/2}$ convergence rate. The numerical studies described here indicate that in *most* shielding problems, the OCF is a better estimator than the TLP; yet in the cases of a unidirectional or nearly parallel beam source with point detectors in its vicinity, the TLP seems to provide a new and more efficient option.

APPENDIX A

GENERALIZATION OF THE TRACK LENGTH ESTIMATION

Let the target response be given in the form

$$I = \int g(\mathbf{x}) \psi(\mathbf{x}) \chi_v(\mathbf{x}) \, d\mathbf{x} \quad (\text{A.1})$$

where

$\psi(\mathbf{x})$ = collision density at phase space point $\mathbf{x}$

$g(\mathbf{x})$ = detector response function

$\chi_v(\mathbf{x})$ = volume function being unity inside a prespecified volume and zero outside.

In order to get a Monte Carlo estimate $\tilde{I}$, of I , the well-known collision estimator can be used given by

TABLE II

Comparison of the Neutron Flux Obtained by TLP and OCF from a Unidirectional Source

Detector Position	Flux by TLP (n/cm ² ·s)	Flux by OCF (n/cm ² ·s)	PRSD TLP (%)	PRSD OCF (%)	q TLP	q OCF	$B = \frac{q_{OCF}}{q_{TLP}}$	Flux by UCF (n/cm ² ·s)
(X, Y, Z)								
(0, 0, 0)	6.27×10^{-2}	6.37×10^{-2}	4.5	5.4	108	216	2	6.24×10^{-2}
(0, 0, 1)	3.42×10^{-1}	3.34×10^{-1}	1.62	2.4	14	41.9	2.99	3.32×10^{-2}
(0, 0, 3)	2.52×10^{-1}	2.33×10^{-1}	1.29	2.42	8.9	43	4.8	2.52×10^{-1}
(0, 0, 10)	7.13×10^{-2}	4.48×10^{-2}	3.3	7.9	58.2	463	7.9	6.23×10^{-2}
(0, 0, 20)	1.07×10^{-2}	7.26×10^{-3}	9.0	19	433	2675	6.1	1.21×10^{-2}
(0, 0, 30)	1.52×10^{-3}	1.23×10^{-3}	14.6	34.7	1149	8922	7.7	2.04×10^{-3}
(0, 4.9, 30)	4.17×10^{-5}	4.19×10^{-5}	13.1	12.1	930	1086	1.17	4.24×10^{-5}

$$\eta(\mathbf{x}_1, \dots, \mathbf{x}_m) = \sum_{i=1}^m g(\mathbf{x}_i) \chi_v(\mathbf{x}_i) \quad (\text{A.2})$$

We now introduce into the transport equation a fictitious scattering cross section $\Sigma_{s\delta}$ such that when the particle enters a fictitious collision, it continues its flight with its direction and energy unaltered.^b With this the particle flux $\phi(\mathbf{x})$ is unchanged. However, the collision rate will change and be given now by

$$\psi^*(\mathbf{x}) = \frac{\Sigma_t^*(\mathbf{x})}{\Sigma_t(\mathbf{x})} \psi(\mathbf{x}) \quad (\text{A.3})$$

where $\Sigma_t^* = \Sigma_t + \Sigma_{s\delta}$ is the modified total cross section. The collision estimator is also slightly modified, taking the form

$$\eta^* = \sum_{i=1}^m \frac{\Sigma_t(\mathbf{x}_i)}{\Sigma_t^*(\mathbf{x}_i)} g(\mathbf{x}_i) \chi_v(\mathbf{x}_i) \quad (\text{A.4})$$

This is easily seen, since one can write

$$I = \int g(\mathbf{x}) \frac{\Sigma_t(\mathbf{x})}{\Sigma_t^*(\mathbf{x})} \psi^*(\mathbf{x}) \chi_v(\mathbf{x}) d\mathbf{x} \quad (\text{A.5})$$

We may now ask what will be the form of the estimator of Eq. (A.4) as $\Sigma_{s\delta} \rightarrow \infty$. Consider the situation where a particle travels a distance d along a straight line of flight in the volume. If the particle makes n fictitious collisions along this path, its contribution to η^* will be^c

$$\alpha_k^*(d) = \sum_{j=1}^K \frac{\Sigma_t(x_j)}{\Sigma_t^*(x_j)} g(x_j) \begin{cases} k = n \text{ if path ends on the boundary surface} \\ k = n + 1 \text{ if path ends at real collision point} \end{cases}$$

Let us first find out what will be the conditional average of $\alpha_k^*(d)$, given that the particle made exactly n fictitious collisions along the distance d . Consider the conditional pdf $f(x_1, \dots, x_n | n, d)$ for the n fictitious collisions to occur at a specific sequence of points $(x_1, x_2, \dots, x_n)$ along the straight line of length d , given that exactly n such collisions occurred. The probability that n fictitious collisions occur along d is given by

$$P(n|d) = \frac{(\Sigma_{s\delta} \cdot d)^n}{n!} \exp(-\Sigma_{s\delta} \cdot d) \quad (\text{A.6})$$

This can easily be deduced by induction, since $P(n|d)$ and $P(n-1|d)$ maintain the recurrence relation

$$P(n|d) = \int_0^d P(n-1|x) \exp[-\Sigma_{s\delta}(d-x)] \Sigma_{s\delta} dx,$$

and since

$$P(0|d) = \exp(-\Sigma_{s\delta} \cdot d).$$

Furthermore, the unconditional pdf for n fictitious collisions to occur at $(x_1, \dots, x_n)$ is given by

$$\begin{aligned} f(x_1, \dots, x_n) dx_1 \dots dx_n \\ = \exp(-\Sigma_{s\delta} \cdot x_1) \Sigma_{s\delta} dx_1 \\ \times \exp[-\Sigma_{s\delta}(x_2 - x_1)] \Sigma_{s\delta} dx_2 \dots \\ \exp[-\Sigma_{s\delta}(x_n - x_{n-1})] \Sigma_{s\delta} dx_n \exp[-\Sigma_{s\delta}(d - x_n)] \\ = \Sigma_{s\delta}^n \exp(-\Sigma_{s\delta} \cdot d) dx_1 \dots dx_n, \end{aligned}$$

assuming, of course, that $\Sigma_{s\delta}$ is constant. The conditional pdf can now be easily derived from the relation

$$P(n|d) \cdot f(x_1, \dots, x_n | n, d) = f(x_1, \dots, x_n)$$

^bThis derivation follows, up to Eq. (A.6), the derivation of the track length estimator in Ref. 8. There, however, the track length estimator is derived only for the case of reaction rate calculation, where $g(x) = \Sigma_r/\Sigma_t$ and is constant along the straight line of flight of the particle, whereas here we assume no restrictions on $g(x)$. Specifically, it may vary along the flight path.

^cIn the following, we use the simplified notation x_i to denote the point $\mathbf{x}_i = \mathbf{r} + x_i \cdot \boldsymbol{\Omega}$ where $\mathbf{r}$ is the origin of the line and $\boldsymbol{\Omega}$ its direction.

yielding

$$f(x_1, \dots, x_n | n, d) = \frac{n!}{d^n}.$$

Denoting $g^*(\mathbf{x}) = [\Sigma_t(\mathbf{x})/\Sigma_t^*(\mathbf{x})]g(\mathbf{x})$, we can now calculate the conditional average of $\alpha_k^*(d)$ as

$$\begin{aligned} \langle \alpha^* | n \rangle &= \int_0^d dx_1 \int_{x_1}^d dx_2 \dots \int_{x_{n-1}}^d f(x_1, \dots, x_n | n, d) \\ &\quad \times \left[\sum_{j=1}^K g^*(x_j) \right] dx_n \\ &= \frac{n!}{d^n} \int_0^d dx_1 \int_{x_1}^d dx_2 \dots \int_{x_{n-1}}^d \\ &\quad \times \left[\sum_{j=1}^K g^*(x_j) \right] dx_n. \end{aligned} \quad (\text{A.7a})$$

When $k = n$, we first integrate over x_{j+1} up to x_n , obtaining

$$\begin{aligned} \langle \alpha^* | n \rangle &= \sum_{j=1}^n \frac{n!}{d^n} \int_0^d dx_1 \int_{x_1}^d dx_2 \dots \\ &\quad \int_{x_{j-1}}^d \frac{g^*(x_j)(d-x_j)^{n-j}}{(n-j)!} dx_j. \end{aligned} \quad (\text{A.7b})$$

Note that in Eq. (A.7a), the averaging is done by integrating over $x_1, \dots, x_j$, maintaining the relation $x_1 < x_2 < \dots < x_{j-1} < x_j$, which is achieved by integrating each x_i from x_{i-1} to d (and x_1 from 0 to d). The same average can be obtained by reversing the limits of integration; i.e., x_1 is integrated from 0 to x_2 , x_2 from 0 to x_3 , and in general, x_i is integrated between the limits $[0, x_{i+1}]$ and x_j in the limits $(0, d)$. Thus we can write

$$\begin{aligned} \langle \alpha^* | n \rangle &= \sum_{j=1}^n \frac{n!}{d^n} \int_0^d \frac{g^*(x_j)(d-x_j)^{n-j}}{(n-j)!} dx_j \\ &\quad \times \int_0^{x_j} dx_{j-1} \dots \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \\ &= \sum_{j=1}^n \frac{n!}{d^n} \int_0^d g^*(x_j) \\ &\quad \times \frac{(d-x_j)^{(n-j)}}{(n-j)!} \frac{x_j^{j-1}}{(j-1)!} dx_j. \end{aligned} \quad (\text{A.7c})$$

Although the change in the order and limits of integration leading from Eq. (A.7b) to (A.7c) is plausible on statistical averaging grounds, it can be proved formally, and a general proof is given in Appendix B.

Since we now have only one variable x_j , we can replace it by x , then change the order of summation, and by integrating obtain

$$\begin{aligned} \langle \alpha^* | n \rangle &= \frac{n!}{d^n(n-1)!} \int_0^d g^*(x) \\ &\quad \times \sum_{j=1}^n \binom{n-1}{j-1} (d-x)^{(n-j)} x^{(j-1)} dx \\ &= \frac{n}{d} \int_0^d g^*(x) dx. \end{aligned} \quad (\text{A.7d})$$

The integration is carried along the straight line from the starting point of the particle (emission from a collision or a boundary) to the boundary of the volume. If the path ends at a collision point $\mathbf{x}_c$, then $k = n + 1$ and a term $g^*(\mathbf{x}_c)$ is added at the real collision site. Thus

$$\begin{aligned} \langle \alpha^* | n \rangle &= \frac{n}{d} \int_0^d g^*(x) dx + \beta, \\ \beta &= \begin{cases} 0 & \text{—path ends at boundary} \\ g^*(x_c) & \text{—path ends at collision site } \mathbf{x}_c \end{cases}. \end{aligned} \quad (\text{A.8})$$

The unconditional average value of $\alpha_k^*(d)$ over a path of length d will now be given as

$$\langle \alpha^* | d \rangle = \sum_{n=0}^{\infty} P(n|d) \langle \alpha^* | n \rangle,$$

where $P(n|d)$ is given in Eq. (A.6).

Now using Eqs. (A.6) and (A.8), we obtain

$$\begin{aligned} \langle \alpha^* | d \rangle &= \int_0^d g^*(x) dx \sum_{n=0}^{\infty} \frac{n}{d} \frac{(\Sigma_{s\delta} \cdot d)^n}{n!} \exp(-\Sigma_{s\delta} \cdot d) \\ &\quad + \beta \sum_{n=0}^{\infty} P(n|d) \\ &= \Sigma_{s\delta} \cdot \int_0^d \frac{\Sigma_t}{\Sigma_t^*} g(x) dx + \beta. \end{aligned}$$

We can now take the limit of $\langle \alpha^* | d \rangle$ as $\Sigma_{s\delta} \rightarrow \infty$, and since, for every value of $\Sigma_{s\delta}$, $\langle \alpha^* | d \rangle$ is an average of a valid unbiased estimator of I , so will be its limit. Thus we obtain the limit

$$\lim_{\Sigma_{s\delta} \rightarrow \infty} \langle \alpha^* | d \rangle = \int_0^d \Sigma_t(x) g(x) dx = G(d). \quad (\text{A.9})$$

The above result of Eq. (A.9) can be concluded in the following general statement: Supposing $g(\mathbf{x})$ to be any detector response, then a general track length estimator for the estimation of

$$I = \int g(\mathbf{x}) \psi(\mathbf{x}) d\mathbf{x}$$

is given by

$$\eta_l = \sum_{i=1}^m G(d_i) ,$$

where $d_1, \dots, d_m$ are the straight paths of a given history in the detector volume.

APPENDIX B

Theorem: Let $g(x)$ be defined in the interval $[0, d]$ and be integrated in it. Then,

$$\begin{aligned} \int_0^d dx_1 \int_{x_1}^d dx_2 \dots \int_{x_{n-2}}^d dx_{n-1} \int_{x_{n-1}}^d g(x_n) dx_n \\ = \int_0^d \frac{x^{n-1}}{(n-1)!} g(x) dx . \end{aligned} \quad (B.1)$$

Proof: This theorem is most conveniently proven by induction. Consider the case $n = 2$. For that case the theorem reads:

$$\int_0^d dx_1 \int_{x_1}^d g(x_2) dx_2 = \int_0^d x g(x) dx . \quad (B.2)$$

Let $G(x)$ be the primitive function of $g(x)$, then the left side of Eq. (B.1) yields:

$$\begin{aligned} \int_0^d dx_1 \int_{x_1}^d g(x_2) dx_2 &= \int_0^d dx_1 [G(d) - G(x_1)] \\ &= dG(d) - \int_0^d G(x) dx . \end{aligned}$$

The right side yields, through integration by parts,

$$\int_0^d x g(x) dx = dG(d) - \int_0^d G(x) dx ,$$

and the identity Eq. (B.2) is proven.

Assume now that the theorem is true for $n = k$, and consider the case $n = k + 1$. Denoting

$$g^1(x) = \int_x^d g(y) dy ,$$

we get

$$\begin{aligned} I &= \int_0^d dx_1 \int_{x_1}^d dx_2 \dots \int_{x_{n-1}}^d dx_n \int_{x_n}^d g(x_{n+1}) dx_{n+1} \\ &= \int_0^d dx_1 \dots \int_{x_n}^d g^1(x_n) dx_n , \end{aligned}$$

and finally using the induction assumption, we obtain

$$\begin{aligned} I &= \int_0^d \frac{x^{n-1}}{(n-1)!} g^1(x) dx \\ &= \int_0^d \frac{x^{n-1}}{(n-1)!} [G(d) - G(x)] dx \\ &= \frac{d^n}{n!} G(d) - \int_0^d \frac{x^{n-1}}{(n-1)!} G(x) dx \\ &= \int_0^d \frac{x^n}{n!} g(x) dx . \end{aligned} \quad (B.3)$$

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