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On the Markov Chain Analysis of Source Iteration Monte Carlo Procedures for Criticality Problems: II

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Abstract—*Recently a Markov chain process corresponding to a source iteration procedure was proposed and analyzed. The properties of this process are further analyzed, yielding an integral equation for the first-order bias in the eigenmode and an explicit first-order expression for the bias in the eigenvalue.*

Sur l'analyse de l'itération de source à l'aide de la chaîne de Markoff—Techniques de Monte Carlo pour traiter les problèmes de criticalité: II

Résumé—*On a proposé et analysé récemment un procédé basé sur la chaîne de Markoff qui correspond à une technique d'itération de source. Les propriétés de ce procédé sont analysées plus profondément et on dérive une équation intégrale pour le biais de premier ordre du mode propre et, en première approximation, une expression explicite pour le biais de la valeur propre.*

Zur Markow-Ketten-Analyse der Quelliteration—Monte-Carlo-Verfahren bei Kritikalitätsproblemen: II

Zusammenfassung—*Vor kurzem wurde ein Markow-Ketten-Prozeß, der einem Quelliterationsverfahren entspricht, vorgeschlagen und diskutiert. Die Eigenschaften dieses Prozesses werden näher untersucht, und es ergibt sich eine Integralgleichung für den systematischen Fehler erster Ordnung bei der Eigenfunktion sowie, in erster Annäherung, ein expliziter Ausdruck für den systematischen Fehler des Eigenwerts.*

INTRODUCTION

Monte Carlo methods have been successfully applied for many years for both fixed-source problems (shielding systems) and problems involving critical systems. In shielding problems, a solid and rigorous theoretical base exists. That is, the Markov chain process governing the calculation is well defined and was thoroughly analyzed to show that unbiased estimates of functionals of the flux result with a variety of estimators. Basically, this means that when one follows particles through the medium with a random walk process based on the transport kernel, the distribution obtained of the collision points correctly represents, apart from the statistical fluctuations, the collision rate which is the solution of the fixed-source stationary transport equation. The analysis of fixed-source problems was relatively easy due to the fact that the transport kernel fulfills certain subcritical conditions[1] that ensure the convergence of the Neumann series of the flux.

The situation for critical systems is substantially different and considerably more difficult. Since the kernel of the eigenvalue transport equation is critical, the main mode could not be presented as a converging Neumann series. A theoretical analysis of the convergence properties of the Markov process governing Monte Carlo calculations of critical systems did not exist. The procedures devised were primarily based on the intuitive notion that a computer simulation of a mathematical procedure closely resembling a critical system should yield, after enough generations, a fission rate distribution correctly representing the main eigenmode of the critical transport system.

It has long been known that the source iteration method, which is the most common procedure, yields biased results. That is, the distribution of the fission sites for a finite number of particles in each generation differs from the main eigenmode even when the number of generations is increased to infinity. This bias of course yields a bias in every functional of the eigenmode and particularly in the eigenvalue λ_0. Due to the fact that no theoretical account of the convergence of the calculation was reported, very little is known about the nature and magnitude of this bias. Accumulated practical experience indicated that for most critical systems the bias in λ_0 is <2%. Attempts were made to study the nature of the bias by systematic numerical studies comparing results of Monte Carlo calculations and deterministic calculations.[2] There is also evidence indicating that the bias depends on $1/N$ where N is the number of particles in each generation.

Recently a detailed analysis of a Markov chain model of a Monte Carlo procedure for criticality problems was proposed.[3] In this analysis, it was proven that the Monte Carlo estimate of λ_0 is asymptotically unbiased and that if we denote by λ_N^n the estimate of λ_0 at the n'th generation, where N particles are used in each generation, and by

$$\lambda_N = \lim_{n\to\infty} \lambda_N^n \ ,$$

then

$$|\lambda_N - \lambda_0| \leq \frac{C}{\sqrt{N}} \ . \tag{1}$$

This inequality of course means that as $N \to \infty$, $\lambda_N \to \lambda_0$, yet it is of very little practical value, as far as the estimation of the bias is concerned, because the constant C is prohibitively difficult to calculate. Furthermore, it is a weak inequality in the sense that it does not support the assumed N^{-1} convergence rate (although it is true for every value of N).

In the following, we attempt to shed some more light on the problem. Starting from the model of Ref. 3, we concentrate on the distributions of the fission sites, showing first that the fission site distribution asymptotically converges to the eigenmode. This is done by deriving the equation fulfilled by that distribution when $n \to \infty$ and showing that the equation differs from the transport equation for any finite N, yet it approaches the transport equation as $N \to \infty$. Further, we analyze the asymptotic behavior of the equation and, using first-order perturbations, we obtain an estimate of the bias in λ_0, which may be of practical value since it is easy to calculate and follows an N^{-1} behavior.

We first bring some general definitions of the criticality problem and briefly review the results of Ref. 3 that will be used henceforth.

The critical system is represented by the eigenvalue transport equation

$$\lambda_0 \Phi_0(R) = \int K(P|R)\Phi_0(P)\, dP \ , \tag{2}$$

where R and P are phase space points and $K(P|R)$ is the fission kernel that can be defined as the density number of neutrons that will enter fission at R due to one neutron entering fission at P (per unit phase space volume). The value $\Phi_0(R)$ is the main eigenmode. It is the number density of neutrons entering fission at R and we can with no loss of generality assume that $\Phi_0(R)$ is normalized to unity. Thus,

$$\int \Phi_0(R)\, dR = 1 \ . \tag{3}$$

The eigenvalue is λ_0.

If we now denote by

$$K(P) = \int K(P|R)\, dR \ , \tag{4}$$

then $K(P)$ is the expected number of fissions that will result all over phase space from one neutron entering fission at P. It is then easy to see that

$$\lambda_0 = \int K(P)\Phi_0(P)\, dP \ . \tag{5}$$

With this equation, it is possible to interpret λ_0 as the expected number of fissions resulting from a fission source (of neutrons entering fission) with distribution $\Phi_0(R)$. The target of criticality calculations is to obtain the distribution $\Phi_0(R)$ of fission sites and functionals of it such as Eq. (5).

The source iteration procedure described in Ref. 3 has the following features. We start with the zeroth generation of N points $(P_1, \ldots, P_N)$ and proceed according to the algorithm

1. Set $k = 1$.

2. Start a particle at point P_i chosen with probability

$$p_s(P_j) = \frac{\nu(P_j)}{\sum_{j=1}^{N} \nu(P_j)} \ ,$$

   where $\nu(P)$ is the average number of fission neutrons produced in a fission at P.

3. Follow the particle in the medium until it is terminated by either fission, absorption, or leakage.

4. If the particle is terminated by absorption or leakage, go to step 2, and restart with a new particle (with, maybe, a new starting point from the set $(P_1, \ldots, P_N)$.

5. If the particle is terminated by fission at a point R, then set $R_k = R$ and then $k = k + 1$.
6. If $k = N + 1$, stop.
7. If $k < N + 1$, go to step 2.

It is easy to see that this algorithm means that we repeatedly start particles from the set $(P_1, \ldots, P_N)$ until we obtain a new set of N fission sites $(R_1^1, \ldots, R_N^1)$. The set $(R_1^1, \ldots, R_N^1)$ is the first generation. To proceed, we now use the first generation to replace $(P_1, \ldots, P_N)$ and again follow the above algorithm and produce the second generation $(R_1^2, R_2^2, \ldots, R_N^2) = \bar{R}_N^2$. The process is repeated and a sequence of generations $\bar{R}_N^1 \to \bar{R}_N^2 \to \ldots \to \bar{R}_N^n$ is so obtained. This type of procedure with various modifications is the basic feature of the source iteration method. Intuitively, one would hope that as n increases the set of fission points $(R_1^n, \ldots, R_N^n)$ would be a sample from $\Phi_0(R)$ such that if $a(R)$ is some function of phase space, then the estimator

$$\eta_N^n(a) = \frac{1}{N} \sum_{i=1}^{N} a(R_i^N) \tag{6}$$

has the property that

$$\eta_N(a) = \lim_{n \to \infty} \eta_N^n(a)$$

is an unbiased estimator of the functional

$$I = \int \Phi_0(R) a(R)\, dR \ . \tag{7}$$

If we take $a(R) = K(R)$ of Eq. (4), then $\eta_N(K)$ will be an unbiased estimator of λ_0. In that case we will say that the procedure is unbiased and the estimate $\eta_N(a)$ that we will obtain for I will have only the usual uncertainty represented with the standard deviation resulting from the finite sample of size N used.

We then see that the question of bias in the procedure amounts to the question whether the set $(R_1^n, \ldots, R_N^n)$ as $n \to \infty$ is a sample from $\Phi_0(R)$ or not. This is thus the main problem on which one should concentrate in analyzing the bias. It is already well known that a bias that depends on N exists. To try and see the features of that bias, we will next attempt to investigate, through the properties of the Markov chain on which the procedure is based, the relation between the set $(R_1^n, \ldots, R_N^n)$ as $n \to \infty$ and $\Phi_0(R)$. This is the meeting point between the Monte Carlo procedure and the critical transport process represented by Eq. (2).

THE LIMITING DISTRIBUTION OF $\bar{R}_N^n$

The transfer probability from the set $\bar{R}_N^n$ to the state $\bar{R}_N^{n+1}$ is given by[3]

$$K(R_1^n, \ldots, R_N^n | R_1^{n+1}, \ldots, R_N^{n+1}) = \prod_{l=1}^{n} \left[\frac{\sum_{i=1}^{N} K(R_i^n | R_l^{n+1})}{\sum_{i=1}^{N} K(R_i^n)} \right] , \tag{8}$$

where $K(\bar{R}_N^n | \bar{R}_N^{n+1})$ is the transfer probability of the Markov process. From the general theory of Markov processes, it is known that the distribution of the set $(\bar{R}_N^n)$ as n approaches infinity approaches a steady-state distribution $\rho_s(R_1, \ldots, R_N)$, which is defined by the equation[3]

$$\rho_s(R_1, \ldots, R_N) = \int \ldots \int K(P_1, \ldots, P_N | R_1, \ldots, R_N) \times \rho_s(P_1, \ldots, P_N)\, dP_1 \ldots dP_N \ , \tag{9}$$

where $\rho_s(R_1, \ldots, R_N)$ is symmetrical in its N variables in the sense that it is invariant to any permutation of its variables. This is obvious for physical reasons and can also be seen from Eqs. (8) and (9). This symmetry means that the marginal distributions of each R_i are identical, i.e.,

$$\rho_i(R) = \int \int \rho_s(R_1, \ldots, R_N) \delta(R_i - R)\, dR_1 \ldots dR_N = \rho_j(R) = \rho_N(R) \text{ for any } i \text{ and } j \ . \tag{10}$$

Because of this symmetry, sampling N points $(R_1, \ldots, R_N)$ from $\rho_s(\bar{R}_N)$ is identical to sampling N times from the marginal distribution $\rho_N(R)$. Thus, the N fission points obtained in the process as $n \to \infty$ can be viewed as a sample of size N from the function $\rho_N(R)$. If we now recall that in order for the process to be unbiased it is required that $(R_1, \ldots, R_N)$ be a sample from $\Phi_0(R)$, then it is clear that it is required that $\rho_N(R) = \Phi_0(R)$. Thus, if the marginal single variable distribution of the limiting stationary distribution of the Markov process differs from the main eigenmode, the procedure is biased. From Eqs. (8), (9), and (10), we can get

$$\begin{aligned}\rho_N(R) &= \int \ldots \int K(P_1, \ldots, P_N | R_1, \ldots, R_N) \\ &\quad \times \rho_s(P_1, \ldots, P_N) \delta(R_i - R) \\ &\quad \times dP_1 \ldots dP_N dR_1 \ldots dR_N \\ &= \int \ldots \int \rho_s(P_1, \ldots, P_N) \\ &\quad \times \prod_{l=1}^{N} \frac{\sum_{j=1}^{N} K(P_j | R_l)}{\sum_{j=1}^{N} K(P_j)} \\ &\quad \times \delta(R_i - R)\, dP_1 \ldots dP_N dR_1 \ldots dR_N \ . \end{aligned} \tag{11}$$

Noting that for every $l \neq 1$ the integration over R_l yields a multiplicative unity in the integrand, we may conclude that

$$\rho_N(R) = \int \ldots \int \rho_s(P_1, \ldots, P_N) \times \frac{\left[\frac{1}{N}\sum_{j=1}^{N} K(P_j|R)\right]}{\left[\frac{1}{N}\sum_{j=1}^{N} K(P_j)\right]} dP_1 \ldots dP_N \ , \quad (12)$$

where we have also divided the sums in the numerator and the denominator by N. Since sampling a set $(P_1, \ldots, P_N)$ from $\rho_s(P_1, \ldots, P_N)$ is equivalent to sampling N times from $\rho_N(P)$, the sums can be viewed as sample averages of the random variables $K(P_i|R)$ and $K(P_i)$. Consequently, as N approaches infinity, the sample averages will converge in probability to their expected values, namely,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} K(P_j|R) = \int \rho(P)K(P|R)\, dR \quad (13)$$

and

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} K(P_j) = \int K(P)\rho(P)\, dP = \lambda \ , \quad (14)$$

where

$$\rho(P) = \lim_{N\to\infty} \rho_N(P) \ .$$

Thus, in the limit as $N \to \infty$ the sums are independent of $(P_1, \ldots, P_N)$, and since $\rho_s(P_1, \ldots, P_N)$ is normalized to unity for every N, we may conclude that

$$\lim_{N\to\infty} \rho_N(R) = \rho(R) = \frac{\int \rho(P)K(P|R)\, dP}{\lambda} \ . \quad (15)$$

Now, comparing Eq. (15) with Eq. (2), we see that $\rho(R) = \Phi_0(R)$ and $\lambda = \lambda_0$. Thus, the process is asymptotically unbiased in the sense that as the number of generations goes to infinity and the number of particles in each generation (N) approaches infinity, the fission sites will be distributed according to the eigenmode $\Phi_0(R)$. Yet for a finite number of particles the marginal distribution $\rho_N(R)$ fulfills Eq. (12), which is different from the transport equation, and thus the process is biased. In order to get an idea about the amount of bias for a finite N, it is necessary to look in more detail at the rate of convergence of $\rho_N(R)$ of Eq. (12) to $\Phi_0(R)$ of Eq. (2). Yet, before doing so, it may be worthwhile to give an example of Monte Carlo procedure similar to the one discussed above, which does not even asymptotically converge to the eigenmode. Recall that in the above procedure if a particle is not terminated by fission, a new particle is started from a *new* starting point chosen with probability $p_s(P_j)$ (see step 2 of the algorithm). Let us now consider a Markov process in which once a starting point is chosen particles are started from *that point* repeatedly until fission is obtained. This procedure is quite close to the one discussed above. One may easily verify that the equation of the marginal distribution for this case will be

$$\rho'_N(R) = \int \ldots \int \rho'_s(P_1, \ldots, P_N) \times \frac{\frac{1}{N}\sum_{j=1}^{N} \frac{\nu(P_j)K(P_j|R)}{K(P_j)}}{\frac{1}{N}\sum_{j=1}^{N} \nu(P_j)} dP_1 \ldots dP_N \ . \quad (16)$$

As $N \to \infty$, Eq. (16) will yield, using the same arguments as for (15),

$$\lim_{N\to\infty} \rho'_N(R) = \rho'(R) = \int \frac{\nu(P)K(P|R)}{\lambda' K(P)} \rho'(P)\, dP \ ,$$

$$\lambda' = \int \nu(P)\rho'(P)\, dP \ . \quad (17)$$

Equation (17) differs from Eq. (2), and thus the process does not converge to the eigenmode. Note that only if $\nu(P)/K(P) =$ constant will Eq. (17) coincide with Eq. (2). That condition actually means that the probability of fission is the same for a particle starting at any point in phase space. We brought in the above example for two reasons:

1. to stress the point that not every Monte Carlo process that seems to simulate the critical system will yield asymptotically unbiased results, and small changes in an unbiased process may render it biased
2. to indicate that the above method of constructing the Markov chain and observing the convergence of the limiting equation for the marginal distribution may be suitable to check the validity of Monte Carlo procedures for critical systems.

FIRST-ORDER PERTURBATION ESTIMATE FOR THE BIAS

Consider again Eq. (12). It can be written as

$$\rho_N(R) = \sum_{j=1}^{N} \int \ldots \int \rho_s(P_1, \ldots, P_N) \times \frac{K(P_j|R)}{\sum_{i=1}^{N} K(P_i)} dP_1 \ldots dP_N \ . \quad (18)$$

Since $\rho_s(P_1, \ldots, P_N)$ is symmetrical, the right side is the sum of N identical functions independent of j. Thus writing $\bar{P}_N = (P, P_1, \ldots, P_{N-1})$, ρ_N may be written as:

$$\rho_N(R) = N\int \ldots \int \rho_s(P, P_1, \ldots, P_{N-1}) \times \frac{K(P|R)}{K(P) + \sum_{i=1}^{N-1} K(P_i)} \, dPdP_1 \ldots dP_{N-1}$$

$$= \int \ldots \int \rho_s(P, P_1, \ldots, P_{N-1}) \times \frac{K(P|R)}{\left[\frac{1}{N}K(P) + \frac{1}{N}\sum_{i=1}^{N-1} K(P_i)\right]} \times dPdP_1 \ldots dP_{N-1}$$

$$= \int \ldots \int \rho_s(P, P_1, \ldots, P_{N-1}) \times \frac{K(P|R)}{\left[\frac{1}{N}K(P) + \frac{N-1}{N} \cdot \frac{1}{N-1}\sum_{i=1}^{N-1} K(P_i)\right]} \times dPdP_1 \ldots dP_{N-1} \ , \tag{19}$$

where in Eq. (19) we multiplied and divided by $N - 1$.

Let us look at the asymptotic behavior of this integral. From Eq. (14) we know that the sample average in the denominator converges to λ_0 as $N - 1 \to \infty$. Thus, we assume that $N - 1$ is large enough so that the sample average can be replaced by λ_0, yielding

$$\rho_N(R) = \int \ldots \int \rho_s(P, P_1, \ldots, P_{N-1}) \times \frac{K(P|R)}{\lambda_0\left[1 + \frac{K(P)}{N\lambda_0} - \frac{1}{N}\right]} \, dPdP_1 \ldots dP_{N-1} \ , \tag{20}$$

or, using Eq. (10):

$$\rho_N(R) = \int \rho_N(P) \frac{K(P|R)}{\lambda_0\left[1 + \frac{K(P)}{N\lambda_0} - \frac{1}{N}\right]} \, dP \ . \tag{21}$$

Equation (21) is the asymptotic equation for $\rho_N(R)$. It differs from the transport equation by the term $[K(P)/N\lambda_0 - 1/N]$ in the denominator and clearly converges to Eq. (2) as $N \to \infty$.

We see that for any finite N the source iteration method does not comply with the exact eigenvalue equation. The above analysis reveals that although the source iteration method seems, intuitively, to correctly simulate the physical situation, still, even in the asymptotic case when the number of generations is taken to infinity and the number of fission points is large, the distribution of the fission points is distorted. The ability to quantitatively see the difference between $\rho_N(R)$ and $\phi_0(R)$ as reflected in Eqs. (2) and (21) seems to be a major achievement of the Markov chain approach.

We now denote by $\Delta\rho(R)$ the difference between $\rho_N(R)$ and $\phi_0(R)$, i.e., $\rho_N(R) = \phi_0(R) + \Delta\rho(R)$. Substitution into Eq. (21) yields

$$\phi_0(R) + \Delta\rho(R) = \int \Delta\rho(P)K'(P|R)\, dP + \int \phi_o(P)K'(P|R)\, dP \ , \tag{22}$$

where

$$K'(P|R) = \frac{K(P|R)}{\lambda_0\left[1 + \frac{K(P)}{N\lambda_0} - \frac{1}{N}\right]} \ .$$

Equation (22) may be written as a nonhomogeneous equation of the form $H\Delta\rho = S(R)$, with

$$H\Delta\rho = \Delta\rho(R) - \int \Delta\rho(P)K'(P|R)\, dP$$

and

$$S(R) = \int \phi_0(P)K'(P|R)\, dP - \phi_0(R) \ .$$

Let $\psi^*(R)$ denote the solution of the adjoint homogeneous equation $H^*\psi^* = 0$. Then, in order that Eq. (22) will have a solution, it is necessary and sufficient that

$$\psi^*(R)S(R)dR = 0 \ .$$

However,

$$\psi^*(R)S(R)dR = \int\int \psi^*(R)[\phi_0(P)K'(P|R)\, dP - \phi_0(R)]\, dR \ ,$$

and using the fact that $\psi^*(R)K'(P|R)dR = \psi^*(P)$, we obtain

$$\psi^*(R)S(R)dR = \int \psi^*(P)\phi_0(P)\, dP - \int \psi^*(R)\phi_0(R)\, dR \equiv 0 \ ,$$

and thus Eq. (22) has a solution $\Delta\rho(R)$.

We now proceed to obtain a first-order estimate for the bias in K_{eff}. Denoting $a(P) = [K(P)/\lambda_0 - 1]$, the denominator of Eq. (21) can be expanded as

$$\rho_N(R) = \int \rho_N(P) \frac{K(P|R)}{\lambda_0} \times \left[1 - \frac{a(P)}{N} + \frac{a^2(P)}{N^2} - \frac{a^3(P)}{N^3} + \ldots\right] dP$$
$$= \int \rho_N(P) \frac{K(P|R)}{\lambda_0} dP + \int \rho_N(P) \frac{K(P|R)}{\lambda_0} \times \left[\sum_{n=1}^{\infty} \frac{(-1)^n a^n(P)}{n^n}\right] dP \tag{23}$$

or

$$\phi_0(R) + \Delta\rho(R) = \int \phi_0(P) \frac{K(P|R)}{\lambda_0} dP + \int \Delta\rho(P) \frac{K(P|R)}{\lambda_0} dP + \int \frac{\Delta\rho(P) K(P|R)}{\lambda_0} \left[\sum_{n=1}^{\infty} (-1)^n \frac{a^n(P)}{N^n}\right] dP + \int \phi_0(P) \frac{K(P|R)}{\lambda_0} \left[\sum (-1)^n \frac{a^n(P)}{N^n}\right] dP , \tag{24}$$

and using Eq. (2) we obtain:

$$\Delta\rho(R) = \int \frac{\Delta\rho(P) K(P|R)}{\lambda_0} dP + \int dP \frac{\Delta\rho_0(P) K(P|R)}{\lambda_0} \times \left[\sum_{n=1}^{\infty} (-1)^n \frac{a^n(P)}{N^n}\right] + \int \frac{\phi_0(P) K(P|R)}{\lambda_0} \cdot \left[\sum_{n=1}^{\infty} (-1)^n \frac{a^n(P)}{N^n}\right] dP . \tag{25}$$

Integrating both sides over R yields zero on the left side because both $\rho_N(R)$ and $\phi_0(R)$ are probability density functions (pdf's) normalized to unity thus:

$$\int \Delta\rho(R)\, dR \equiv 0 .$$

Rearranging the terms we obtain:

$$\int \Delta\rho(P) K(P)\, dP = \int \phi_0(P) K(P) \sum_{n=1}^{\infty} (-1)^{n+1} \times \frac{[K(P) - \lambda_0]^n}{\lambda_0^n N^n} dP + \int \Delta\rho(P) K(P) \sum_{n=1}^{\infty} (-1)^{n+1} \times \frac{[K(P) - \lambda_0]^n}{\lambda_0^n n^n} dP . \tag{26}$$

Recalling now Eqs. (6) and (7), we may note that for every response $D(R)$, the Monte Carlo calculation will yield an estimate of

$$I' = \int \rho_N(R) D(R)\, dR = \int \phi_0(R) D(R)\, dR + \int \Delta\rho(R) D(R)\, dR = I + \Delta I .$$

Thus the bias will be given by

$$\Delta I = \int \Delta\rho(R) D(R)\, dR .$$

For the very important case of the calculation of λ_0, we note that $D(R) = K(R)$, since

$$\lambda_0 = \int K(P) \phi_0(P)\, dP .$$

We thus see that the left side of Eq. (26), is the bias in the eigenvalue. Looking at the right side of Eq. (26), we note that a first-order approximation may be obtained by taking only the $n = 1$ term in the first expression of the right side. This yields

$$\Delta\lambda = \int \frac{\phi_0(P) K(P) [K(P) - \lambda_0]}{\lambda_0 N} dP . \tag{27}$$

All other terms will have a higher order of convergence in N. This is so because

$$\lim_{N\to\infty} \Delta\rho(R) = 0$$

and thus the $n = 1$ term in the second expression will converge to zero faster than N^{-1}. Thus Eq. (27) represents a first-order approximation to the bias in the eigenvalue.

A few more points are worth mentioning. Notice that both Eqs. (27) and (26) are consistent with the physical observation that in the case of an infinite homogeneous medium the bias should vanish. In that case, $K(P) \equiv \lambda_0$ and the right sides of both Eqs. (27) and (26) vanish. Equation (27) represents the fact that the bias depends on $1/N$. This was observed quite often in the past in practical calculations. Equation (27) has a practical value, since all the quantities involved may be quite easily estimated for a given problem. Equation (27) also has a simple statistical interpretation: Looking at $K(P)$ as an estimator of λ_0 with $\phi_0(P)$ being the pdf for P, it is readily seen that the right side is simply the variance of $K(P)$ since

$$\Delta\lambda = \frac{\int \phi_0(P) K^2(P)\, dP - \lambda_0 \int \phi_0(P) K(P)\, dP}{\lambda_0 N} = \frac{\langle K^2(P)\rangle - \langle K(P)\rangle^2}{\lambda_0 N} . \tag{28}$$

CONCLUSIONS

A Monte Carlo procedure for criticality problems was analyzed through the limiting properties of the Markov chain governing the random walk. The relation between the marginal distribution of the fission sites and the eigenmode was discussed, showing that this relation is decisive in the validity of the source iteration process. It was shown that the source iteration procedure with a repeated choice of starting point to obtain N next-generation fission sites is asymptotically unbiased, and an explicit first-order expression for the bias in λ_0 is derived. Also, an integral equation for the bias in the eigenmode is obtained.

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