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Levitation and oscillations of neutral particles in a constant electric field

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We investigate oscillations of polarized particles embedded in a host liquid and subjected to a constant electric field. In the considered range of parameters there exist repulsion forces between particles which can cause particle levitation and oscillations in the system. We study the eigenfrequencies of oscillations of the levitated particle and demonstrate that oscillations frequency decreases when the amplitude of the electric field grows. Consequently, the lower frequency cut-off of the excited oscillations is determined by the maximum strength of the external electric field whereby the particle motion is not overdamped. The upper bound of the oscillations frequency is determined by the minimum strength of the electric field which is required for particle levitation. We analyze two simple geometrical configurations when a spherical particle is levitated by another immobilized spherical particle and the particle is levitated above the plane interface between two half-spaces with different permittivity and electric conductivity. © *2011 American Institute of Physics*. [doi:10.1063/1.3592183]

I. INTRODUCTION

One of the methods for determining properties and parameters of particles involves levitation of the polarized particles by the external electric field.[1] Levitation of particle in different media and various geometries was extensively investigated in the past.[2–6] The goal of these studies was elucidating the conditions of stability of levitation[3–6] and effect of the geometry of electrodes. Theoretical analysis in the most of the previous investigations is based on the effective dipole model. Using this model the formal criterion for stable levitation was determined in.[3] Extension of the analysis of forces acting at the particles beyond the framework of dipole approximation, whereby higher order terms in the multipole expansion with respect to the centers of the particles are accounted for, was performed in Ref. 4. The approach adopted in the latter study is valid only in the case of an ideal dielectric. In the present study, we employ a more general approach for determining forces and torques acting at the particles which is not restricted by a dipole approximation and assumption of an ideal dielectric. This approach allows us to derive a closed system of equations for determining forces acting at the particles in different regimes.

When the equilibrium position of a levitated particle is stable, the particle returns at the equilibrium position after a small shift from the equilibrium, and we can speak about eigenoscillations of the particle. Measuring the frequency of these oscillations provides additional information which can be used for determining parameters of the system. In spite of the large number of publications on levitation of neutral polarized particles by the external electric field, there are only a few studies, e.g., Ref. 6 which investigate particle oscillations. Complicated geometry of the system which was considered in Ref. 6 does not allow deriving simple formulas for the dependency of the eigenfrequency on the amplitude of the external electric field. In the present study, we analyze two simple geometrical configurations when (i) a spherical particle is levitated due to the repulsion force caused by another immobilized spherical particle, and (ii) levitation of the spherical particle caused by repulsion from the plane interface between two half-spaces with different permittivity and electric conductivity. The external homogeneous electric field causes polarization of the particles whereby the levitated particle and the repulsive particle are the sources of the inhomogeneous electric fields caused by polarization.

A general feature of the considered oscillations is a peculiar dependence of particle oscillation frequency on the amplitude of the external electric field. In particular, the oscillation frequency decreases when the amplitude of the external electric field grows. Consequently, the minimum frequency of oscillations, which are excited with the amplitude growth, is determined by the condition of overdamped motion in the presence of finite viscosity, while the maximum oscillation frequency corresponds to the minimum magnitude of the electric field which is required to compensate the gravitation force.

It must be emphasized that in the framework of the model with piecewise-smooth medium, without taking into account explicitly the contact between particles, the particles remain electrically neutral, i.e., the total electric charge of each particle is equal zero. The distribution of the electric charge over the particle surface does not exhibit monopole symmetry and it is determined by the symmetry of the electric fields. This is exactly the case that is considered in the present study. In Sec. II, we describe a mathematical model and outline the general procedure for determining forces in the framework of the model with piecewise-smooth medium. This model is used in Sec. III for calculating eigenfrequency of the particle which is levitated by other immobilized particle. In Sec. IV, we consider a case when a spherical particle that is embedded in a host fluid is levitated above the plane

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interface between the host fluid and a semi-infinite domain having electric properties different from those of a host fluid. In Appendix A and Appendix B we present general formulas for taking into account high order terms in the multipole expansion and determine the conditions when these higher order terms can be neglected.

II. MATHEMATICAL MODEL

The set of equations describing a weakly conducting system was suggested in Ref. 7. This general approach is presented below in order to make the paper self-contained and consistent.

In a system with a finite electric conductivity, electric charge accumulates at the particle surface due to two phenomena: i) electric polarization which develops on microscopic time scale that is much smaller than other characteristic time scales in the problem and is considered negligibly small, and ii) transport of the electric charge due to the finite electric conductivity. The need to account for the finite relaxation time of the electric charge constitutes the main difference between the weakly conducting system and the ideal dielectric.

For solving the problem, it is necessary to determine the electric potential $\Phi = \Phi(\vec{r}, t)$ that determines all electrodynamic parameters of the system. The equations describing the electrodynamic part of the problem reads[7–9]

$$\vec{\nabla} \cdot \vec{D} = \rho_{ex}, \tag{1}$$

$$\frac{\partial \rho_{ex}}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0, \tag{2}$$

$$\vec{D} = \varepsilon_0 \varepsilon \vec{E}, \quad \vec{j} = \vec{j}_\sigma + \vec{j}_c, \quad \vec{j}_\sigma = \sigma \vec{E}, \quad \vec{E} = -\nabla \Phi, \tag{3}$$

where ρ_{ex} is a free charge that is formed during relaxation process [see Eq. (2)], ε, σ are local permittivity and electric conductivity of the medium, $\vec{j}_c$ is a convective electric current caused by a macroscopic motion of the charge, $\vec{j}_c = \rho_{ex} \vec{v}$, and $\vec{v}$ is a macroscopic velocity of the electric charge.

In the system with piecewise-constant physical properties, electric charges accumulated at the boundaries between the media with different electric parameters. Therefore, the above equations imply that $\int \rho_{ex} dV = \int \gamma dS$ or $\rho_{ex} = \gamma \delta(u) |\vec{\nabla} u|$, where $\delta(u)$ is a Dirac's delta function, $u = F(x, y, z, t)$ and $u = 0$ is an equation of the particle surface. Since it is assumed that macroscopic motion of the electric charge is caused by the motion of a particle surface, $\vec{v} = \dot{\vec{s}}(t) + \vec{\Omega} \times \vec{r}_s$, where $\vec{r}_s$ are coordinates of the surface and $\vec{s}(t)$ is the location of the center of mass of the particle. Taking into account that $div\ \vec{v} = 0$, and $\partial u / \partial t + \vec{v} \cdot \vec{\nabla} u = 0$ we obtain the following boundary conditions at the particle-host medium interface:

$$\left[\vec{n} \cdot \vec{D} \right] = \gamma, \quad \left[\vec{n} \cdot \vec{j}_\sigma \right] = -\frac{\partial \gamma}{\partial t} - \vec{v} \cdot \nabla \gamma, \tag{4}$$

where $[A] = A_+ - A_-$, A_+ and A_- are the values of the function A at the external and at the internal surfaces, respectively, $\vec{n}$ is the external unit normal vector at the particle surface.

Forces acting at the particles are caused by the interaction of the particle charges with the external electric field. In the framework of the piece-wise smooth model, the total charge of the i-th particle is determined by the jump of the electric field strength at the particle surface[9,10]

$$\gamma_T^i = \varepsilon_0 \vec{n} \cdot \left[\vec{E} \right]^i, \tag{5}$$

while the external electric field acting at the particle is determined by the continuous at the particle surface component of the electric field strength. Consequently, the expression for the force acting at the particle reads:

$$\vec{F}^i = \int \gamma_T^i (\vec{E}_0 + \vec{E}_c^i) ds, \tag{6}$$

where $\vec{E}_0$ is an electric field caused by the external source in the absence of particles and $\vec{E}_c^i$ is an electric field at the surface of the i-th particle which is induced by other particles. The formula for torque that acts at the particle reads

$$\vec{M} = \int \gamma_T \left[\vec{r}(s) \times (\vec{E}_c + \vec{E}_0) \right] ds.$$

The above analysis implies that the electric potential in the vicinity of the i-th particle, $\Phi(r, t)$, can be written as follows:

$$\Phi = \Phi^i = \Phi_s^i + \Phi_0 + \Phi_c^i, \tag{7}$$

where the derivatives of the functions Φ_s^i are discontinuous at the particle surfaces, Φ_0 is a potential induced by the external source in the absence of particles, and functions Φ_c^i are continuous together with their derivatives at the particle surfaces and describe the contribution of the additional particles

$$\Phi_s^i = \Phi_{in}^i \theta_i + \Phi_{out}^i (1 - \theta_i), \quad \Phi_c^i = \Phi_R^i + \sum_{j \neq i} \left(\Phi_{out}^j + \Phi_R^j \right), \tag{8}$$

where $\theta_i = 1$ inside the domain occupied by the i-th particle and $\theta_i = 0$ outside this domain, Φ_R^i and Φ_R^j are perturbations of the potential of the i-th and j-th particle caused by electrodes. Each of the potentials $\Phi_{in}^i, \Phi_{out}^i, \Phi_R^i$ satisfies the Laplace equation $\vec{\nabla}^2 \Phi = 0$ with the boundary conditions which follow from the boundary conditions (4). At the surface of the i-th particle

$$\varepsilon_0 \vec{n} \cdot (\varepsilon_{out} \vec{\nabla} \Phi_{out}^i - \varepsilon_i \vec{\nabla} \Phi_{in}^i) = \varepsilon_0 (\varepsilon_i - \varepsilon_{out}) \vec{n} \cdot \vec{\nabla} (\Phi_0 + \Phi_c^i) - \gamma^i, \tag{9}$$

$$\vec{n} \cdot (\sigma_{out} \vec{\nabla} \Phi_{out}^i - \sigma_i \vec{\nabla} \Phi_{in}^i) = (\sigma_i - \sigma_{out}) \vec{n} \cdot \vec{\nabla} (\Phi_0 + \Phi_c^i) + \dot{\gamma}^i + \vec{v}^i \cdot \vec{\nabla} \gamma^i, \tag{10}$$

where γ^i is the free electric charge at the surface of the i-th particle which was determined above, $\vec{v}^i$ is a velocity of the i-th particle and $\varepsilon_i, \sigma_i, \varepsilon_{out}, \sigma_{out}$ permittivity and conductivity of the particle and of the host medium, respectively. Boundary conditions (9), (10) must be supplemented by the conditions of continuity of a potential at the particle surface and conditions of constant potential at the plane ideal electrodes. For $z = 0$, $\Phi = \Phi_0(0) = 0$, and for $z = d$, $\Phi = \Phi_0(d) = -E_0 d$. These conditions at the particle surface imply that $\Phi^i_{in} = \Phi^i_{out}$, while at the electrodes surface

$$\Phi^i_R + \Phi^i_{out} = 0. \tag{11}$$

It should be emphasized that we consider a case when a surface charge accumulated by the particle is induced by the external field. If the particle was initially charged independently of the applied electric field, then this charge is included additively in the scheme. Formulas (6) include, as a particular case, the dipole approximation which has been used extensively for the analysis of the weakly-conducting systems.

Equation (6) yields the following expression for the force acting at the i-th particle:

$$\vec{F^i} = \varepsilon_0 \int \left[\vec{n} \cdot \left(\vec{\nabla}\Phi^i_{out} - \vec{\nabla}\Phi^i_{in}\right)\right]\vec{\nabla}\left(\Phi_0 + \Phi^i_c\right) ds_i. \tag{12}$$

The outlined above approach does not depend upon the geometry of particles, and this approach is applicable to any medium with piece-wise constant properties comprising the particles with a finite size.

Hereafter, we consider a case of spherical particles with radii a_i and with the centers, $[0, 0, s_i(t)]$, located at the straight line which is perpendicular to the electrodes. The external electric field Φ_0 is assumed homogeneous. We seek for the solution in the form of the expansions into the eigenfunctions $Y^i_{\ell,in}$ and $Y^i_{\ell,out}$ of the Laplace equation in the internal and in the external domains corresponding to the i-th sphere

$$\begin{pmatrix} Y^i_{\ell,in} \\ Y^i_{\ell,out} \end{pmatrix} = P_\ell(\mu_i)\begin{pmatrix} u_i^\ell \\ u_i^{-\ell-1} \end{pmatrix}, \qquad u_i = |\vec{r} - \vec{s}_i|/a_i,$$
$$\mu_i = [(\vec{r} - \vec{s}_i) \cdot \vec{e}_3]/|\vec{r} - \vec{s}_i|,$$

where $\vec{e}_3$ is a unit vector in the positive direction of z-axis, $P_\ell(x)$ are Legendre polynomials, $\vec{s}_i = s_i\vec{e}_3$,

$$\Phi_0 = -E_0 z, \quad \begin{pmatrix} \Phi^i_{in} \\ \Phi^i_{out} \end{pmatrix} = -E_0 a_i \sum_{\ell=1} c^i_\ell(t)\begin{pmatrix} Y^i_{\ell,in} \\ Y^i_{\ell,out} \end{pmatrix}, \tag{13}$$

$$\Phi^i_R = \sum_{\ell=1}^{\infty} \Phi^i_{R,\ell},$$
$$\Phi^i_{R,\ell} = -E_0 a_i c^i_\ell \left[\sum_{N=-\infty}^{\infty} (-1)^{\ell+1} \bar{Y}^{i,N}_{\ell,out} + \sum_{N\neq 0,-\infty}^{\infty} Y^{i,N}_{\ell,out}\right], \tag{14}$$

where

$$\begin{pmatrix} Y^{i,N}_{\ell,out} \\ \bar{Y}^{i,N}_{\ell,out} \end{pmatrix} = \begin{bmatrix} P_\ell(\mu_{iN})/(u_{iN})^{\ell+1} \\ P_\ell(\mu^+_{iN})/(u^+_{iN})^{\ell+1} \end{bmatrix}, \tag{15}$$

u_{iN}, u^+_{iN} are absolute values of the vectors $\vec{u}_{iN} = [\vec{r} - (2dN + s_i) \cdot \vec{e}_3]/a_i$ and $\vec{u}^+_{iN} = [\vec{r} + (2dN + s_i) \cdot \vec{e}_3]/a_i$. The absolute values of these two vectors, $u_{iN} = |\vec{r} - (2dN + s_i)\vec{e}_3|/a_i$ and $u^+_{iN} = |\vec{r} + (2dN + s_i)\vec{e}_3|/a_i$ satisfy the following conditions: $u_{iN} = u^+_{iN}$ for the plane $z = 0$ and $u_{iN+1} = u^+_{iN}$ for the plane $z = d$. The angular parameters, $\mu_{iN} = \vec{u}_{iN} \cdot \vec{e}_3/u_{iN}$ and $\mu^+_{iN} = \vec{u}^+_{iN} \cdot \vec{e}_3/u^+_{iN}$, satisfy the following relations: $\mu_{iN} = -\mu^+_{iN}$ for $z = 0$ and $\mu_{iN+1} = -\mu^+_{iN}$ for $z = d$. Taking into account these relations, it can be easily shown that $\Phi^i_R + \Phi^i_{out} = 0$ for $z = 0$ and $z = d$. Therefore, the solution of the problem is reduced to determining the coefficients c^i_ℓ using the boundary conditions (9), (10). In order to formulate the equations which determine these coefficients we use the relation between solutions in spherical coordinate systems with different origins.[11] In particular, we use the relation between a function Y_{out} determined in the coordinate system with the origin in $\vec{r}_B = (0, 0, s_B)$ and a function Y_{in} determined in the coordinate system with the origin in $\vec{r}_A = (0, 0, s_A)$

$$\frac{P_n(\mu_B)}{|\vec{r} - \vec{r}_B|^{n+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k P_k(\mu_A)|r - \vec{r}_A|^k}{|\vec{r}_A - \vec{r}_B|^{n+k+1}} \times \begin{pmatrix} n+k \\ k \end{pmatrix}\{sign[(\vec{r}_A - \vec{r}_B) \cdot \vec{e}_3]\}^{n+k}, \tag{16}$$

Equation (16) allows us to introduce matrix $T^{ij}_{m\ell}$ (for details see Appendix A) and to determine expressions for different functions Y_{out} in the formulas for Φ^i_c [see Eq. (8)] in the vicinity of the i-th particle in the form of expansion into functions $Y^i_{m,in}$

$$\Phi^i_c = -E_0 a_i \sum_{i\neq j} \sum_{\ell=1}^{\infty} \sum_{m=0}^{\infty} Y^i_{m,in} T^{ij}_{m\ell} c^j_\ell, \tag{17}$$

Substituting Eq. (17) in the boundary condition (9) yields a system of linear equations for determining the coefficients c^i_ℓ for a given magnitude of a free charge

$$[(\ell+1)/\ell + \kappa_{i\varepsilon} + 1]E_0(t)\ell c^i_\ell + \kappa^i_\varepsilon \ell T^{ij}_{\ell k} E_0(t) c^j_k = -\kappa^i_\varepsilon \delta_{\ell 1} E_0(t) - \gamma^i_\ell/\varepsilon_0\varepsilon_{out}, \tag{18}$$

where $\gamma^i_\ell = \int_{-1}^1 P_\ell(x)\gamma^i(x)dx$ and $\kappa^i_\varepsilon = \varepsilon_i/\varepsilon_{out} - 1$. In Eq. (17) and in Eq. (19) below we imply summation over repeating indices.

In the case of a conductor with a finite electric conductivity, the charge distribution is not conserved and must be determined in the process of solution. Substituting Eq. (17) in the boundary condition (10) and taking into account that

$$\left(\frac{\partial}{\partial t} + \dot{\vec{s}}_i \cdot \nabla\right)\begin{pmatrix} Y^i_{\ell,in} \\ Y^i_{\ell,out} \end{pmatrix} = 0,$$

and assuming the rotationless particles we arrive at the following set of equations:

$$[(\ell+1)/\ell + \kappa_{i\sigma} + 1]E_0(t)\ell c^i_\ell + \kappa^i_\sigma \ell T^{ij}_{\ell k} E_0(t) c^j_k = -\kappa^i_\sigma \delta_{\ell 1} E_0(t) + \dot{\gamma}^i_\ell/\sigma_{out}, \tag{19}$$

where $\kappa^i_\sigma = \sigma_i/\sigma_{out} - 1$.

The system of Eqs. (18) and (19) determines, for a given magnitude of the external electric field $E_0(t)$, $P \times L$ unknown coefficients c_ℓ^i and $P \times L$ "free" charges γ_ℓ^i, where P is a number of particles located between the electrodes and L is the maximum order of the multipoles which depends upon the required accuracy of calculations. Hereafter, we consider the case of the noncharged particles, i.e., it is assumed that $\int \gamma_T^i ds = 0$, and an electric field is determined by Laplace equation with corresponding boundary conditions. Taking into account recharging and formation of charged particles as a results of electrochemical processes, is a separate problem whereby effects which are analyzed in the present study can be of secondary importance. In this study, we assume that the particles are not charged and their interaction is determined by the potential governed by the Laplace equation, i.e., there is no volumetric charge in the system. The presented above analysis is pertinent to this situation. In case of ideal electrodes and initially electro neutral spherical particles, there is no monopole symmetry with respect to the centers of the particles and the particles remain electro neutral provided that the electric field is determined by the Laplace equation.

Let us determine the forces acting at the particles. Taking into account the condition of electro neutrality and substituting Eqs. (13)–(15), (17) in Eq. (12) yields:

$$\vec{F}_i = \sum_{j \neq i} \vec{F}_{ij}, \quad \vec{F}_{ij} = -\vec{e}_3 \varepsilon_0 4\pi E_0^2 a_i^2 \sum_{m,s} (m+1) T_{m+1s}^{ij} c_m^i c_s^j. \tag{20}$$

Formulas (18)–(20) and the expressions for $T_{\ell m}^{ij}$ (see Appendix A) which depend only on the distance between particles and their distance to the electrodes, together with equations of motion of particles provide the complete solution of the problem. Depending on the characteristic spatial and time scales in the problem, Eqs. (18) and (19) allow different approximate solutions. In the zeroth approximation with respect to the matrix coefficients $T_{k\ell}^{ij}$, this system of equations describes the electrodynamics of a single particle that performs a translational motion in the homogeneous electric field. In this approximation charge distribution γ_ℓ^i does not depend upon coordinates of the particles, and Eqs. (17) and (18) can be solved independently of the equations of motion. Another case when Eqs. (17) and (18) can be solved independently of the equations of motion is when the characteristic times of electric charge relaxation are significantly smaller than the characteristic times of the mechanical motion of particles. The characteristic times of relaxation of the electric charge τ_ℓ can be determined by solving Eqs. (18) and (19) with $T_{k\ell}^{ij} = 0$. The characteristic times τ_ℓ weakly depend on the multipole rank of the field, ℓ, and the condition for time separation between the mechanical motion and electric charges relaxation is $\tau_m^i \gg \tau_0 = \varepsilon_0 \varepsilon_{out} / \sigma_{out}$, where τ_m^i are the characteristic times of mechanical motions of particles. Hereafter it is assumed that this condition is satisfied.

III. REPULSION BETWEEN PARTICLES AND OSCILLATIONS OF THE LEVITATED PARTICLE

Forces acting at the particles can be classified according to three mechanisms which induce these forces: (i) direct interactions between particles, $\vec{F}_p^i$, which are caused by polarization of particles in the external electric field; (ii) forces of interactions of particles with their own images, $\vec{F}_{Rs}^i$; (iii) interactions of particles with images of other particles, $\vec{F}_{Rf}^i$. The total force acting at the i-th particle is as follows:

$$\vec{F}^i = \vec{F}_{Rs}^i + \vec{F}_{Rf}^i + \vec{F}_p^i.$$

Strictly speaking, such classification can be performed only in the first order approximation with respect to the matrix T, when the coefficients c_ℓ^i are determined in the zeroth approximation with respect to $T_{k\ell}^{ij}$. In this approximation, the coefficients c_ℓ^i are independent of the distance to the electrodes and of the distance between particles. For the homogeneous external field and in the limit $t \ll \tau_0$ and $t \gg \tau_0$, where $1/\tau_0 = \sigma_{out}/\varepsilon_0 \varepsilon_{out}$, $c_\ell^i = c_0^i \delta_{\ell 1}$ and $c_\ell^i = c_\infty^i \delta_{\ell 1}$, respectively, c_0^i and c_∞^i are determined by the following formulas:

$$c_0^i = -\frac{\kappa_\varepsilon^i}{3 + \kappa_\varepsilon^i}, c_\infty^i = -\frac{\kappa_\sigma^i}{3 + \kappa_\sigma^i}. \tag{21}$$

The explicit expressions for the forces $\vec{F}_{Rf}^i$, and $\vec{F}_{Rs}^i$ can be obtained using the above formulas (see also Appendix A). Hereafter, it is assumed that the particles are located at the sufficiently large distances from the electrodes such that $(a_i/s_i)^3 \ll 1, \quad a_i/(d - s_i) \ll 1$, and all these forces can be neglected. Under these circumstances, the interactions between particles are determined by $\vec{F}_p^i$

$$\vec{F}_p^i = -\vec{e}_3 \varepsilon_0 \, 24\pi E_0^2 \, a_i^3 \, c^i \sum_{j \neq i} c^j sign(s_i - s_j) \frac{a_j^3}{(s_i - s_j)^4}. \tag{22}$$

Formula (22) for interaction forces between particles corresponds to the main dipole approximation which includes only dipole corrections in all orders of the perturbation theory (see Appendix B). Depending on the sign of $\alpha = c^i c^j$, the inter particle force can be either attractive or repulsive. When $\alpha > 0$ the particles are attracted while for $\alpha < 0$ they repel each other.

In Appendix B, we present a general solution of the equations for the coefficients c^i. Inspection of these solutions shows that with the accuracy of corrections, which do not affect the qualitative behavior of the particles, one can use expressions (21) for c^i. Then the expressions for the force of interaction between two particles read

$$\vec{F}^1 = -\vec{F}_2 = -\vec{e}_3 sign(s_1 - s_2) \varepsilon_0 \alpha_0 24\pi E_0^2 \frac{a_1^3 a_2^3}{(s_1 - s_2)^4}, \tag{23}$$

where $\alpha_0 = \kappa_1 \kappa_2 / [(\kappa_1 + 3)(\kappa_2 + 3)]$ and $\kappa_i = \kappa_{i\varepsilon}, \kappa_{i\sigma}$ for the ideal dielectric and for the stationary regime, respectively. Repulsion between particles results in the emergence of the eigenfrequencies of particle oscillations which depend on particle parameters and the magnitude of the external electric field. The condition, $\alpha_0 < 0$, requires that the parameters of the host medium, $\varepsilon_{out}, \sigma_{out}$, lie in the range between the corresponding parameters of the particles. Let us assume that the straight line that connects the centers of the particles, is perpendicular to the electrodes and is directed along the acceleration of gravity (see Fig. 1). The particle 2 is considered

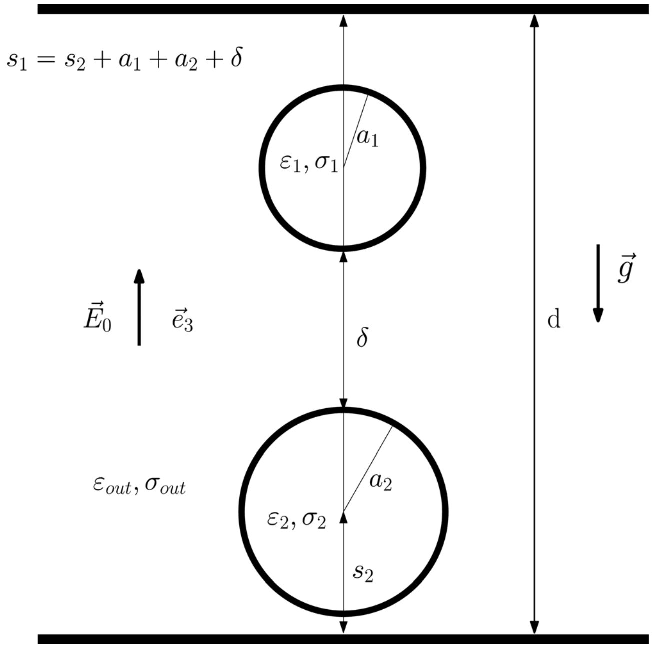

FIG. 1. Two particles in the external electric field produced by two infinite plane electrodes.

immobilized, and the particle 1 can move only in the vertical direction because it is embedded in the host viscous fluid at rest. Taking into account that $s_1 - s_2 = a_1 + a_2 + \delta$, where δ is the distance between the spheres (see Fig. 1), equation of motion of particle 1 in a viscous fluid can be written as follows:

$$\ddot{\delta} = -\bar{g} - \frac{\varepsilon_0 \alpha_0 18 E_0^2 a_2^3}{\rho_1 (a_1 + a_2)^4} \frac{1}{\left(1 + \frac{\delta}{a_1 + a_2}\right)^4} - \frac{9\nu\rho_0}{2\rho_1 a_1^2}\dot{\delta}, \qquad (24)$$

where $\bar{g} = g(1 - \rho_0/\rho_1)$, ρ_1 is the density of the particle, ρ_0 is the density of the host fluid, ν is a kinematic viscosity of the host fluid. The terms in the right-hand-side of Eq. (24) describe the effective weight of the particle, electrophoretic force and viscous force per unit mass of the particle, respectively. It must be noted that we consider the simplest model of a viscous fluid neglecting the apparent mass effect, inertial effects and hydrodynamic interaction between particles. Equation (24) yields the expression for the minimum strength of the external electric field $E = E_c$ required for levitating a particle having effective weight $\bar{g}$. Since the repulsive electrophoretic force [the second term in the right-hand-side of Eq. (24)] attains the maximum for $\delta = 0$, equating the latter value to $\bar{g}$ yields

$$\varepsilon_0 E_c^2 = -\rho_1 \bar{g} \frac{(a_1 + a_2)^4}{18\alpha_0 a_2^3}. \qquad (25)$$

When the strength of the external electric field $E_0 < E_c$, then the electric field cannot compensate the effective particle weight. When $E_0 > E_c$, particle 1 resides in the state of stable equilibrium at the distance δ_0 that can be determined from Eq. (24) and conditions $\ddot{\delta} = 0, \quad \dot{\delta} = 0$. Taking into account Eq. (25) we arrive at the following formula for δ_0:

$$\delta_0 = (a_1 + a_2)\left[\left(\frac{E_0^2}{E_c^2}\right)^{1/4} - 1\right]. \qquad (26)$$

Inspection of Eq. (26) shows that for $E_0 < E_c$ the levitated particle cannot be supported by the electrophoretic force applied by the other particle. In order to determine the eigenfrequency let us rewrite Eq. (24) using the parameter $u = \delta - \delta_0$, i.e., displacement of the coordinate of particle 1 from the equilibrium. Equations (24)–(26) yield:

$$\ddot{u} = -\bar{g} + \bar{g}\frac{E_0^2}{E_c^2} \frac{1}{\left(\sqrt{\frac{E_0}{E_c}} + \frac{u}{a_1 + a_2}\right)^4} - \frac{9\nu\rho_0}{2\rho_1 a_1^2}\dot{u}. \qquad (27)$$

The condition for the absence of particle collisions is $u < \delta_0$. In the case of the inviscid fluid the displacement u is determined by energy conservation:

$$\frac{\dot{u}^2}{2} + \bar{g}u + \frac{\bar{g}E_0^2(a_1 + a_2)}{3E_c^2\left(\sqrt{\frac{E_0}{E_c}} + \frac{u}{(a_2 + a_1)}\right)^3} = const.$$

When $u \ll (a_1 + a_2)\sqrt{E_0/E_c}$, the Eq. (27) can be linearized, and the expression for the particle eigenfrequency reads

$$\omega^2 = \frac{4\bar{g}}{a_1 + a_2}\sqrt{\frac{E_c}{E_0}}. \qquad (28)$$

Therefore, the oscillation frequency decreases when the amplitude of the external electric field grows, and at some amplitude the particle motion becomes overdamped. The maximum magnitude of the eigenfrequency is $\omega^2 = \omega_m^2 = 4\bar{g}/(a_1 + a_2)$.

In the linear region $u(t) \sim e^{\gamma t}$, where $\gamma = -9\bar{\nu}/4a_1^2 \pm\sqrt{81\bar{\nu}^2/16a_1^4 - \omega^2}$ and $\bar{\nu} = \nu\rho_0/\rho_1$. The condition that the motion of the particle is not overdamped, $\mathrm{Im}\,\gamma = 0$, determines the maximum strength of the electric field E_m whereby particle oscillations are still possible (the equilibrium point is a focus). This condition yields

$$\sqrt{\frac{E_m}{E_c}} = \frac{64\bar{g}a_1^4}{81\bar{\nu}^2(a_1 + a_2)}. \qquad (29)$$

Oscillations exist when the strength of the external electric field is in the range $E_c < E < E_m$. The condition $E_m > E_c$ determines the range of the particle radii whereby oscillations can be observed, $a_1^4 > 81\bar{\nu}^2(a_1 + a_2)/64\bar{g}$. When $\bar{\nu} \sim 10^{-1}$ cm^2/s, $\bar{g} \sim 10^3 \mathrm{cm}/s^2$, and $a_2 \sim a_1$, this condition is satisfied for the particles with the size $a_1 > 10^{-5/3}$cm. For particles having this size and $E_m \sim E_c$, the eigenfrequency $\omega \sim 10^2 s^{-1}$. This dynamics is realized in the range of the strength of the electric field which is determined by Eq. (25). When parameter $\bar{g}$ and the particle density $\rho_1 \sim 1g/\mathrm{cm}^3$, $E_c \sim 10^3 - 10^4 V/\mathrm{cm}$.

The obtained results can be readily generalized to the case when $\bar{g} < 0$. To this end, it is sufficient to replace the above formula for $s_1 - s_2$ by $s_1 - s_2 = -a_1 - a_2 - \delta$. The latter formula corresponds to the case when the levitated particle is located below the immobilized particle. Since Eq.

(23) is invariant with respect to the transformation, $\bar{g} \to -\bar{g}, \delta \to -\delta, a_1 + a_2 \to -(a_1 + a_2)$, all other formulas are also invariant with respect to this transformation.

IV. LEVITATION AND OSCILLATIONS OF PARTICLE IN THE VICINITY OF A PLANE INTERFACE

Levitation of the particle considered in the previous section requires special conditions (one of the particles must be immobilized) and cannot occur spontaneously in particle suspension. In order to study feasibility of particle levitation in suspensions, it is quite natural to investigate particle interaction with the walls of the vessel. This raises the question of whether particles can be levitated above a plane surface. The above discussion has hitherto been restricted to the case when a plane surface is an ideal electrode. Let us consider now the case when a plane surface is not an ideal electrode but separates two media with different conductivity and permittivity, $\varepsilon_{out}, \sigma_{out}$, and ε_n, σ_n. A spherical particle with electric parameters ε, σ and having radius a is immersed in the host medium with parameters $\varepsilon_{out}, \sigma_{out}$ (see Fig. 2). In this problem, the presence of other particles is not important and can be accounted for additively. It is assumed also that the particle is located at large distances from the other surfaces which produce the electric field, and the main effect is due to the particle interaction with the interface (the plane $z = 0$). Without a particle, the electric field in the medium denoted by the index *out* is homogeneous and is described by the potential Φ_0, and $\vec{\nabla}\Phi_0 = -\vec{E}_0$. The formal solution of the problem is the same in the case of ideal dielectric and in the stationary regime. For sake of definiteness, let us consider the case of ideal dielectric. The following conditions are satisfied at the interface between two media in the absence of a particle: $\Phi_0 = \Phi_{0n}$, $\varepsilon_{out}\vec{e}_3 \cdot \vec{E}_0 = \varepsilon_n \vec{e}_3 \cdot \vec{E}_{0n}$. If the interface is parallel to the plane ideal electrodes then $\vec{E}_0 = \vec{e}_3\, \varepsilon_n \Delta/\bar{\varepsilon}$, $\vec{E}_{0n} = \vec{e}_3\, \varepsilon_{out}\Delta/\bar{\varepsilon}$, where $\bar{\varepsilon} = \varepsilon_n l_2 + \varepsilon_{out} l_1$, Δ is a potential difference between electrodes, $E_0 l_2 + E_n l_1 = \Delta$. Hereafter, the field $\vec{E}_0$ is considered to be known. Solution in the medium denoted by the index *out* is similar to that given by Eq. (7)

$$\Phi_{out} = \Phi_0 + \Phi_S + \Phi_R.$$

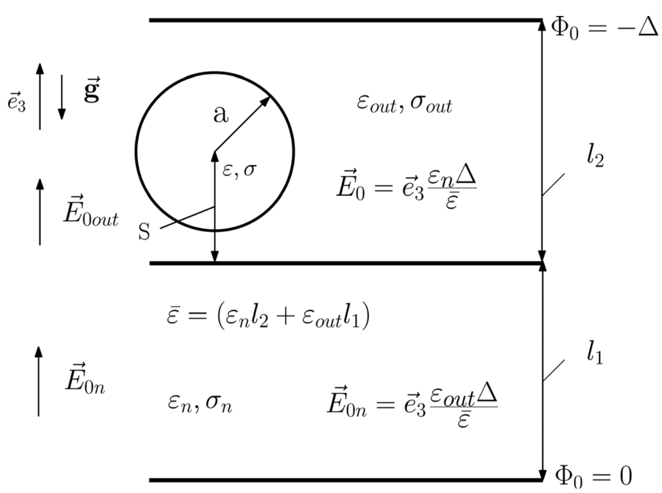

FIG. 2. Levitation of particle near the plane interface between two media.

Assume that the particle is located at $(0, 0, s)$. Then Eqs. (8), (12), (14) yield

$$\Phi_0 = -E_0 z, \quad \Phi_s = \Phi_{in}(\theta) + \Phi_{out}(1-\theta),$$
$$\Phi_R = -E_0 a \sum_{\ell=1}^{n} d_\ell \bar{Y}_{\ell,out},$$

where

$$\begin{pmatrix}\Phi_{in}\\ \Phi_{out}\end{pmatrix} = -E_0 a \sum c_\ell \begin{pmatrix} Y_{\ell,in}\\ Y_{\ell,out}\end{pmatrix},$$
$$\begin{pmatrix}Y_{\ell,in}\\ Y_{\ell,out}\end{pmatrix} = P_\ell(\mu)\begin{pmatrix}u^\ell\\ u^{-(\ell+1)}\end{pmatrix}, \quad \bar{Y}_{\ell,out} = P_\ell(\mu_+)/(u_+)^{\ell+1}, \tag{30}$$

and $u = |\vec{r} - s\vec{e}_3|$, $u_+ = |\vec{r} + s\vec{e}_3|$, $\mu = \vec{e}_3 \cdot (\vec{r} - s\vec{e}_3)/u$, $\mu_+ = \vec{e}_3 \cdot (\vec{r} + s\vec{e}_3)/u^+$.

Potential in the medium denoted by n, $\Phi_n = \Phi_{0n} + \sum f_\ell Y_{\ell,out}$. Boundary conditions at the plane interface imply that $\Phi_{out} = \Phi_n$, $\varepsilon_{out}\vec{e}_3 \cdot \vec{\nabla}\Phi_{out} = \varepsilon_n \vec{e}_3 \cdot \vec{\nabla}\Phi_n$. The system of equations for coefficients c_ℓ, d_ℓ, f_ℓ reads

$$c_\ell + (-1)^\ell d_\ell = f_\ell, \quad \varepsilon_{out}\left(c_\ell + (-1)^{\ell+1} d_\ell\right) = \varepsilon_n f_\ell.$$

Therefore,

$$d_\ell = (-1)^\ell \frac{\varepsilon_{out} - \varepsilon_n}{\varepsilon_{out} + \varepsilon_n} c_\ell, \quad f_\ell = \left(\frac{2\varepsilon_{out}}{\varepsilon_{out} + \varepsilon_n}\right) c_\ell. \tag{31}$$

Substituting Eq. (31) in Eq. (18), that corresponds to the transformation, $\ell T^{ij}_{\ell k} c^j_k \to \ell \bar{T}^i_{\ell k} d_k$, where $\bar{T}^i_{\ell k}$ is determined by Eq. (A-2) and the second equation in Eqs. (A-1), for $N = 0$ we obtain

$$[(\ell+1)/\ell + \kappa_\varepsilon + 1]\ell c_\ell - \kappa_\varepsilon \ell t^+_{\ell k}(0)\beta c_k = -\kappa_\varepsilon \delta_{\ell 1},$$
$$\beta = (\varepsilon_{out} - \varepsilon_n)/(\varepsilon_{out} + \varepsilon_n), \quad \kappa_\varepsilon = \varepsilon_1/\varepsilon_{out} - 1, \tag{32}$$

where expression for $t^+_{\ell k}$ reads

$$t^+_{\ell k}(0) = (-1)^\ell \binom{\ell + k}{k}\left(\frac{a}{2s}\right)^{\ell+k+1} \tag{33}$$

A force acting at the particles near the interface is determined by Eq. (20) with the substitution $T^{ij}_{m+1s} c^j_s \to \bar{T}^i_{m+1s} d_s$

$$\vec{F} = \vec{e}_3 \varepsilon_0 4\pi\beta E_0^2 a^2 \sum_{m,s}(m+1)t^+_{m+1s}(0) c_m c_s. \tag{34}$$

It is possible to use a finite number of ℓ, $1 < \ell < L$, under the condition that $t^+_{\ell k} \to 0$ for $\ell, k \to \infty$. Since $s = a + \delta$, where δ is the distance between the surface of a sphere and the plane $z = 0$, it can be shown that this condition can be satisfied only when $\delta = a(1 + \lambda)$, where $\lambda > 0$. Hereafter, we assume that this condition is satisfied. In the leading dipole approximation $c_\ell = c\delta_{\ell 1}$, and Eq. (34) can be written as follows:

$$\vec{F} = \vec{e}_3 \varepsilon_0 \frac{3\pi}{2}\beta E_0^2 a^2 c^2 \left(\frac{a}{s}\right)^4, \quad c = -\frac{\kappa_\varepsilon}{\kappa_\varepsilon + 3 - \kappa_\varepsilon t^+_{11}\beta}. \tag{35}$$

Equation (35) implies that the direction of the force acting at the particle is determined only by the sign of the parameter β. The case with a stationary regime is recovered by replacing in all formulas $\varepsilon \to \sigma$. In the case of an ideal electrode, $\varepsilon_n \gg \varepsilon_{out}$ ($\sigma_n \gg \sigma_{out}$), $\beta < 0$ and a particle is attracted to an ideal electrode. When $\beta > 0$, the particle is repelled from the surface, and it is possible to excite oscillations in the system. Formally, the analysis of this facet can be performed exactly as in the previous section. Equation (33) implies that $t^{+}_{11} = -(1+\delta/a)^{-3}/4$. Taking into account the indicated above restriction for the values of the parameter δ, it can be showed that with accuracy of the order of $1/32$, $c = -\kappa_\varepsilon/(\kappa_\varepsilon + 3)$. Because the matrix elements, $t^{+}_{11}, t^{+}_{12}, t^{+}_{21}$, are small, accounting for the higher order terms with quadrupole and higher ranks in Eq. (32) has no physical significance and the corresponding formulas are not presented here. Repeating the procedure from the previous section we arrive at the formula for the new critical strength of the electric field, $\bar{E}_c$:

$$\varepsilon_0 \bar{E}_c^2 = \rho_1 \bar{g} \frac{2a}{3\pi\beta\, c^2}.$$

Dynamics of the particle is determined by Eq. (27) with the substitution, $a_1 + a_2 \to a$, and formula for the eigenfrequency reads

$$\omega^2 = \frac{4\bar{g}}{a}\sqrt{\frac{\bar{E}_c}{E_0}}$$

V. CONCLUSIONS

The range of applicability of the suggested in this study approach is quite wide. Using this approach, we investigated dynamics of uncharged particles which are polarized by the external electric field. We considered two cases of particle levitation whereby analytical analysis yields simple formulas for the parameters of particle oscillations. Although the dependence of frequency of oscillations on the amplitude of the external electric field is the same in both cases, the physical mechanisms responsible for particle levitation by the immobilized particle and particle levitation near the interface between two media with different permittivity and conductivity, are completely different.

In the first case particle levitation, in the leading approximation, is caused by interaction by the dipole moment of the levitated particle which is induced by the external homogeneous electric field, with the gradient of the strength of the electric field which is produced by the immobilized repelling particle which is also polarized by the external electric field. Consequently, here the behavior of forces acting at the particle is determined by the difference between electrodynamic parameters of the particle and a host medium as well as by the difference of the electrodynamic parameters of the repelling sphere and a host medium.

In the second case, levitation is caused by self-action of the particle whereby the self-action depends upon the character of polarization of the interface between two media. The direction of the force is determined by the difference between the electrodynamic parameters of the host medium where the particle is located and the electrodynamic parameters of the neighboring medium.

Most of the publications mentioned in the Introduction section investigate direct levitation when the particle is levitated by the external electric field. If coefficient of expansion of the external field potential in the vicinity of some certain location into the Legendre polynomials are known, the problem reduces to that considered in the present study and can be solved using the suggested procedure practically without modifications.

APPENDIX A

In Appendix A, we describe the procedure for calculating the matrix $T^{ij}_{k\ell}$ and derivation of Eq. (17). Taking into account that $u_{iN} = |\vec{r} - \vec{r}_{iN}|/a_i$, $\vec{r}_{iN} = (2dN + s_i)\vec{e}_3$, $u^{+}_{iN} = |\vec{r} - \vec{r}^{+}_{iN}|/a_i$, and $\vec{r}^{+}_{iN} = -(2dN + s_i)\vec{e}_3$, Eqs. (15) and (16) yield:

$$Y^{i,N}_{\ell,out} = \sum_{m=0}^{\infty} Y^{i}_{m,in} t^{i}_{m\ell}(N), \quad \bar{Y}^{i,N}_{\ell,out} = \sum_{m=0}^{\infty} Y^{i}_{m,in} t^{+i}_{m\ell}(N), \tag{A1}$$

where functions $Y^{i,N}_{\ell,out}, \bar{Y}^{i,N}_{\ell,out}$ are determined by Eq. (15), and

$$t^{i}_{m\ell}(N) = (-1)^m \binom{m+\ell}{\ell} \frac{a_i^{m+\ell+1}(-sign(N))^{m+\ell}}{|2dN|^{m+\ell+1}},$$

$$t^{+i}_{m\ell}(N) = (-1)^m \binom{m+\ell}{\ell} \frac{a_i^{m+\ell+1}[sign(2dN + 2s_i)]^{m+\ell}}{|2dN + 2s_i|^{m+\ell+1}}.$$

Let us define the following matrix:

$$\bar{T}^{i}_{m\ell} = \sum_{N\neq 0,-\infty}^{\infty} t^{i}_{m\ell}(N) + \sum_{N=-\infty}^{\infty} (-1)^{\ell+1} t^{+i}_{m\ell}(N). \tag{A2}$$

Using this matrix and Eqs. (A1), (13), (14) we arrive at the following relation:

$$\Phi^{i}_{R} = -E_0 a_i \sum_{m=0}^{\infty}\sum_{\ell=1}^{\infty} Y^{i}_{m,in} \bar{T}^{i}_{m\ell} c^{i}_{\ell}. \tag{A3}$$

The expression for the function Φ^{j}_{R} for $j \neq i$ can be obtained similarly. Taking into account that $u_{jN} = |\vec{r} - \vec{r}_{jN}|/a_j$, $\vec{r}_{jN} = (2dN + s_j)\vec{e}_3$, $u^{+}_{jN} = \left|\vec{r} - \vec{r}^{+}_{jN}\right|/a_j$, $\vec{r}^{+}_{jN} = -(2dN + s_j)\vec{e}_3$, Eqs. (15) can be rewritten as follows:

$$Y^{j,N}_{\ell,out} = \sum_{m=0}^{\infty} Y^{i}_{m,in} t^{ij}_{m\ell}(N), \quad \bar{Y}^{j,N}_{\ell,out} = \sum_{m=0}^{\infty} Y^{i}_{m,in} t^{+ij}_{m\ell}(N),$$

where

$$t^{ij}_{m\ell}(N) = (-1)^m \binom{m+\ell}{\ell} \frac{a_i^m a_j^{\ell+1}\left[sign(s_i - s_j - 2dN)\right]^{m+\ell}}{\left|s_i - s_j - 2dN\right|^{m+\ell+1}}, \tag{A4}$$

$$t^{ij}_{m\ell}(N) = (-1)^m \binom{m+\ell}{\ell} \frac{a_i^m a_j^{\ell+1}\left[sign(s_i - s_j - 2dN)\right]^{m+\ell}}{\left|2dN + s_i + s_j\right|^{m+\ell+1}}. \tag{A5}$$

Similarly to Eq. (A2) let us define the following matrix:

$$\tilde{T}^{ij}_{m\ell} = \sum_{N\neq 0,-\infty}^{\infty} t^{ij}_{m\ell}(N) + \sum_{N=-\infty}^{\infty} (-1)^{\ell+1} t^{+ij}_{m\ell}(N) + t^{ij}_{m\ell}(0). \quad \text{(A6)}$$

The matrix $t^{ij}_{m\ell}(0)$ is determined by Eq. (A4) for $N = 0$. This matrix is written separately since it corresponds to the direct interaction between the i-th and j-th particle while the matrices $t^{ij}_{m\ell}(N)$ and $t^{+ij}_{m\ell}(N)$ correspond to the interaction of the i-th particle with the N-th image of the j-th particle. Using definitions (14), (15) for $j \neq i$ and Eq. (A6) we arrive at the following formula:

$$\Phi^j_R + \Phi^j_{out} = -E_0 a_j \sum_{m=0}^{\infty} \sum_{\ell=1}^{\infty} Y^i_{m,in} \tilde{T}^{ij}_{m\ell} c^j_\ell.$$

Let us introduce the following matrix:

$$T^{ij}_{m\ell} = \bar{T}^i_{m\ell}\delta_{ij} + (1-\delta_{ij})\frac{a_j}{a_i}\tilde{T}^{ij}_{m\ell}, \quad \text{(A7)}$$

where δ_{ij} is Kronecker tensor, $\bar{T}^i_{m\ell}$ is determined by Eq. (A2), $\tilde{T}^{ij}_{m\ell}$ is determined by Eq. (A6). For $\Phi^i_c = \Phi^i_R + \sum_{j\neq i}(\Phi^j_R + \Phi^j_{out})$ formula (A7) implies Eq. (17)

$$\Phi_{ic} = -E_0 a_i \sum_{i\neq j} \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} T^{ij}_{m\ell} u^m_i P_m(\mu_i) c^j_\ell.$$

In order to determine forces acting at the particle in the first nonvanishing approximation with respect to the matrix T, one must insert in Eq. (20) the values of the coefficients c^i_ℓ and c^j_ℓ in the zeroth approximation with respect to this matrix which are given for Eq. (21). The force, $\vec{F}_i$, acting at the particle in this approximation is determined by the following expression:

$$\vec{F}_i = \vec{F}^i_{Rs} + \vec{F}^i_{Rf} + \vec{F}^i_p,$$

where $\vec{F}_{Rs}$ is interaction force between a particle and its reflections that is caused by the presence of ideal electrodes, and $\vec{F}_{Rf}$ is the interaction force between a particle and reflections of other particles caused by the electrodes. The expressions for $\vec{F}_{Rs}$ and $\vec{F}_{Rf}$ read

$$\begin{aligned} \vec{F}^i_{Rs} &= -\vec{e}_3\varepsilon_0\, 4\pi E_0^2\, a_i^2\, 2\left(c^i_1\right)^2 \bar{T}^i_{21}, \\ \vec{F}^i_{Rf} &= -\vec{e}_3\varepsilon_0\, 4\pi E_0^2\, a_i\, 2c^i_1 \sum_{j\neq i} c^j_1 a_j \left[\tilde{T}^{ij}_{21} - t^{ij}_{21}(0)\right], \end{aligned} \quad \text{(A8)}$$

where $\bar{T}^i_{21}$, $\tilde{T}^{ij}_{21}$, t^{ij}_{21} are determined by Eqs. (A2), (A6), and (A4) for $N = 0$ and $m = 2, \ell = 1$.

In the case considered in the present study, $a_{i,j} \ll s_{i,j}, d$ and the forces can be estimated as $\left|\vec{F}^i_{Rs}\right| / \left|\vec{F}^i_p\right| \sim \left|F^i_{Rf}\right| / \left|\vec{F}^i_p\right| \sim (a_2/s_2)^4 \ll 1$. The expression for the force $\vec{F}^i_p$ reads

$$\vec{F}^i_p = -\vec{e}_3\varepsilon_0\, 4\pi E_0^2\, a_i\, 2c^i_1 \sum_{j\neq i} c^j_1 a_j t^{ij}_{21}(0). \quad \text{(A9)}$$

Formula (A9) is valid not only in the first order approximation with respect to matrix T, but also in the leading approximation which takes into account all dipole corrections which are introduced during solving the boundary value problem by successive iterations.

In the case of two particles, substituting equation (A4) for $t^{ij}_{21}(0)$, we arrive at the following formula:

$$\vec{F}^1_p = -\vec{F}^2_p = -\vec{e}_3\varepsilon_0\, 24\pi E_0^2 c^1 c^2\, sign(s_1 - s_2)\frac{a_1^3 a_2^3}{(s_1 - s_2)^4}, \quad \text{(A10)}$$

where in the expressions for the coefficients c^i_ℓ we do not indicate the lower index $\ell = 1$. Using Eq. (A9) and expressions for the coefficients c^i_1 in the leading dipole approximation, in Appendix B we analyze the conditions of the applicability of Eqs. (22)–(23).

APPENDIX B

In Appendix B, we derive an expression for the interaction force between polarized particles in the leading dipole approximation. In this approximation, we calculate the sum of the iteration series whereby at each iteration we keep only the contribution of the dipole component of the electric field. At the same iteration, the contributions of the components having a higher multipole rank is smaller than the contribution of the dipole component.

Consider interaction of two particles which are located at large distances from two ideal electrodes. In the stationary regime $\dot{\gamma} = 0$ in Eq. (19), and in the case of an ideal dielectric $\gamma = 0$ in Eq. (18). Since formal solutions of these two problems are similar one can use any of these equations. Hereafter, we denote κ_ε and κ_σ by κ. The leading dipole approximation, which includes the total sum of the series of dipole corrections in the solution of the problem by iterations, corresponds to the formula $c^i_\ell = \delta_{\ell 1} c^i$. The system of Eqs. (18) or (19) reduces to the system of two equations for the coefficients c^1 and c^2

$$(\kappa_1 + 3)c^1 + \kappa_1 T^{12}_{11} c^2 = -\kappa_1, \quad (\kappa_2 + 3)c^2 + \kappa_2 T^{21}_{11} c^1 = -\kappa_2.$$

Equations (A4), (A6), and (A7) in Appendix A imply that

$$T^{12}_{11} = -2\frac{a_2^3}{|s_1 - s_2|^3}, \quad T^{21}_{11} = -2\frac{a_1^3}{|s_1 - s_2|^3}. \quad \text{(B1)}$$

Using Eq. (B1) the system of two equations for the coefficients c^1 and c^2 can be rewritten as follows:

$$\begin{pmatrix} c^1 \\ c^2 \end{pmatrix} = \frac{1}{\Delta}\begin{bmatrix} -(\kappa_2+3)\kappa_1 + \kappa_1\kappa_2 T^{12}_{11} \\ -(\kappa_1+3)\kappa_2 + \kappa_1\kappa_2 T^{21}_{11} \end{bmatrix}, \quad \Delta = (\kappa_1+3)(\kappa_2+3) \times \left[1 - 4\alpha_0(a_1 a_2)^3/|s_1 - s_2|^6\right], \quad \text{(B2)}$$

where

$$\alpha_0 = \kappa_1\kappa_2/[(\kappa_1+3)(\kappa_2+3)]. \quad \text{(B3)}$$

Since $|s_1 - s_2| = a_1 + a_2 + \delta$, it can be readily shown that the maximum magnitude of the second term in parentheses in the expression for the determinant Δ is attained when

$a_1 = a_2$ and $\delta = 0$, and is equal $\alpha_0/16$. Taking into account that in the whole range of parameters $|\alpha_0| < 1$, hereafter we assume that $\Delta = (\kappa_1 + 3)(\kappa_2 + 3)$. In this approximation, the solution of the equations for the coefficients c^1 and c^2 reads

$$\begin{pmatrix} c^1 \\ c^2 \end{pmatrix} = \begin{pmatrix} c^1_\infty + \alpha_0 T^{12}_{11} \\ c^2_\infty + \alpha_0 T^{21}_{11} \end{pmatrix}, \tag{B4}$$

where

$$\begin{pmatrix} c^1_\infty \\ c^2_\infty \end{pmatrix} = -\begin{pmatrix} \kappa_1/(\kappa_1 + 3) \\ \kappa_2/(\kappa_2 + 3) \end{pmatrix}$$

[see Eq. (21)].

Equations (A10), (B1), (B3) imply that

$$\vec{F}^1_p = -\vec{F}^2_p = -\vec{e}_3 \varepsilon_0\, 24\pi E_0^2\, \alpha_0\, sign(s_1 - s_2)\frac{a_1^3 a_2^3}{(s_1 - s_2)^4}(1 + S), \tag{B5}$$

where $S = 2/|s_1 - s_2|^3\left[a_2^3\kappa_2/(\kappa_2 + 3) + a_1^3\kappa_1/(\kappa_1 + 3)\right]$ and as before in Eq. (B5) we neglected the term $\alpha_0^2 T^{12}_{11} T^{21}_{11} < 1/16$. Analysis of the expression for the parameter S shows that its magnitude either small or positive, $|S| < 1$. Therefore, repulsion corresponds to the condition, $\alpha_0 < 0$, whereby one of the parameters $-1 < \kappa < 0$. Assume that the latter inequality is satisfied for the parameter $\kappa = \kappa_1$. Taking into account the dependence of the distance between two particles on their radii, it can be shown that when the magnitudes of the particle radii are close, the effect of the parameter S can be neglected. When $a_2 \gg a_1$, for small distances between particles, $\delta \ll a_2$, the parameter S varies only weakly and $S \approx 2\kappa_2/(\kappa_2 + 3)$. If $\delta \sim a_2$, $S = 1/4$. In any case this term can be accounted for by introducing a correction factor. Assuming $S = 0$, Eq. (B5) recovers Eq. (23).

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