

Effect of Joule heating on orientation of spheroidal particle in alternating electric field

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We study the change of orientation of a spheroidal particle immersed into a liquid or gaseous host medium under the action of the alternating external electric field. It is assumed that the particle and the host medium have different electric conductivities. We show that the rate of Joule heating of the particle depends on the orientation of its axis of symmetry with respect to the direction of the electric field. If electric conductivity of particle strongly varies with temperature, the Joule heating of the particle affects orientation dynamics and results in the appearance of the new equilibrium orientations. We investigate the dependence of the dynamics of particle rotation and the direction of the equilibrium orientation on the frequency of the electric field. © 2008 American Institute of Physics. [DOI: 10.1063/1.2957078]

I. INTRODUCTION

The dynamics of solid or liquid particles embedded in a host liquid or gaseous medium under the action of an external electric field is of theoretical and technological interest. Technological applications include manipulation of microparticles in biotechnology and genetic engineering, nanotechnology, and noncontact measurements of physical properties of particles. The results obtained in numerous theoretical and experimental studies on particle dynamics under the action of the external electric field were summarized in several survey papers and monographs.¹⁻⁴

In a general case the dynamics of the particle, embedded in a leaky dielectric medium, is quite involved. Most of the studies in this field investigate phenomena associated with Quincke rotation which is caused by a free charge at the particle surface. If the relaxation time of an electric charge inside a body is larger than the relaxation time in the host medium, then for sufficiently large strength of the external electric field, the body begins to rotate around the axis which is perpendicular to the direction of the electric field. This behavior is associated with the peculiarities of charge distribution under the above indicated conditions, and has been discussed in a number of studies.^{2,3,5-7} In this study we analyze another physical effect which was considered in Refs. 8-11. In our previous studies^{10,11} we investigated the dynamics of a spheroidal particle embedded in a leaky dielectric medium under the action of ac electric field. We considered a particle with permittivity ε_2 and conductivity σ_2 which is embedded into a gaseous or liquid host medium with permittivity ε_1 and conductivity σ_1 . It was shown that stable equilibrium orientation (SEO) of the particle depends on the frequency of the applied electric field. In particular, under certain conditions which were established in Ref. 11 [see Eq. (42) in Ref. 11], there exist frequencies ω_1 and ω_2 such that in the frequency range $\omega_1 < \omega < \omega_2$ the direction of the axis

of symmetry of the spheroidal particle is perpendicular to the direction of the same particle in an ideal dielectric system. The additional SEO in a certain frequency range appears due to a finite electric conductivity in the system. The frequencies ω_1 and ω_2 are determined by biquadratic equation [see Eqs. (39), (40), and (41) in Ref. 11]. For $\kappa_e \ll 1$, where $\kappa_e = \varepsilon_2/\varepsilon_1 - 1$, $\omega_2 \rightarrow \infty$, and the frequency ω_1 is determined by a simple formula,

$$\omega_1^2 = \frac{(1+f_z)(1+f_x)}{\tau_0^2}, \quad (1)$$

where $f_z = \kappa_\sigma n$, $f_x = \kappa_\sigma(1-n)/2$, $\kappa_\sigma = \sigma_2/\sigma_1 - 1$, $\tau_0 = \varepsilon_0\varepsilon_1/\sigma_1$, and n is a depolarization coefficient of a spheroid along the axis of symmetry. Hereafter we consider precisely this situation when polarizations of the medium and of the particle can be neglected. In this case we can distinguish between a low-frequency orientation occurring in the frequency range $\omega < \omega_1$ and a high-frequency orientation which is realized for $\omega > \omega_1$. Low-frequency orientation of the particle coincides with the particle orientation in the case of ideal dielectric.

Let θ be the angle between the axis of symmetry and the external electric field vector $\vec{E}$. Then in the low-frequency range, $\omega < \omega_1$, $\theta = 0$ when $n < 1/3$ and $\theta = \pi/2$ when $n > 1/3$. In the high-frequency range, $\omega > \omega_1$, and the same values of depolarization factor n , $\theta = \pi/2$ and $\theta = 0$, respectively.

The physical mechanism of the change in orientation of a spheroidal particle under the action of ac electric field is as follows. Consider, e.g., a prolate spheroid, $n < 1/3$. In the case of an ideal dielectric the principal axis of such spheroid is oriented along the electric field, regardless of the frequency up to the frequency of the order of the inverse time of the microscopic relaxation, τ_m . In a system with a finite electric conductivity due to the finite relaxation time of a free charge, τ_0 , already at significantly smaller frequencies $\omega \ll 1/\tau_m$, the dipole moment follows the direction of the external electric field with delay. During some time interval the direction of the dipole moment is opposite to the direction of

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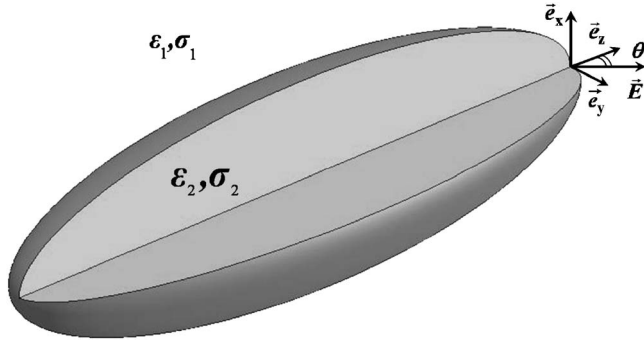


FIG. 1. Ellipsoid with electric permittivity ϵ_2 and conductivity σ_2 inside a host medium with permittivity ϵ_1 and conductivity σ_1 in the external electric field $\vec{E}$.

the field. During this time interval the parallel orientation of the axis of the spheroid is unstable. If the duration of the time interval, whereby the dipole moment is directed in the opposite direction, $T_2(\omega)$ is larger than the time interval $T_1(\omega)$ during which the dipole moment is oriented in the direction of the electric field, the orientation of the principal axis of the particle along the electric field is unstable, and in the limit $t \rightarrow \infty$ the particle will eventually acquire the orientation perpendicular to the external field.¹¹

The frequency threshold ω_1 separating the frequency domains with different orientations of the particle depends on the parameter κ_σ [see Eq. (1)] which depends on the temperature of the particle and the host medium temperature. In the presence of Joule heating the frequency threshold ω_1 , whereby the orientation of the particle flips to the perpendicular orientation, shifts. Since Joule heating rate depends on the orientation, in addition to the orientations with $\theta=0$ and $\theta=\pi/2$, there appear new stable orientations of the particle. The main goal of this study is to elucidate the latter effect, namely, to determine stable orientations of the spheroidal particle in the presence of Joule heating. In the following section we describe a general approach to the problem and then solve the simplified problem which is pertinent to the indicated above range of the parameters.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

Let us consider a spheroidal particle with permittivity ϵ_2 and conductivity σ_2 embedded into a gaseous or liquid host medium with permittivity ϵ_1 and conductivity σ_1 in the homogeneous external electric field with a strength $\vec{E}$. In the frame of reference associated with a particle (see Fig. 1), a unit vector directed along the principal axis of the spheroid is $\vec{e}_z=(0,0,1)$. The components of the external electric field $\vec{E}$ in this frame read

$$\vec{E} = E(t)[- \sin \theta(t), 0, \cos \theta(t)], \quad (2)$$

where $\theta(t)$ is an angle between the particle symmetry axis and a direction of the electric field $\vec{E}$.

Neglecting inertia, evolution of $\theta(t)$ in the presence of the electric field due to the torque $\vec{M}(t)$ acting at the particle is determined by the following equation:

$$\eta_s \frac{d\theta}{dt} = M(t), \quad (3)$$

where $\eta_s d\theta/dt$ is a torque acting at the particle due to the viscous force. In the geometry of our problem.

$$M(t) = \vec{M} \cdot \vec{e}_y, \quad \vec{e}_y = (0, 1, 0), \quad \vec{M} = \epsilon_1 \vec{P}(t) \times \vec{E}, \quad (4)$$

where $\vec{P}(t)$ is a dipole moment of the particle. The strength of the electric field $\vec{E}(t)$ and the dipole moment $\vec{P}(t)$ are determined by the following formulas:

$$\vec{E}(t) = -\nabla\varphi, \quad \vec{P}(t) = \int \gamma_T \vec{r} dS, \quad \gamma_T = \epsilon_0 [\vec{E}] \cdot \vec{N},$$

where $[\vec{E}] \cdot \vec{N}$ denotes the jump of the normal component of the electric field at the particle-host medium boundary. Formulas for the dipole moment $\vec{P}(t)$ and other physical parameters that will be required further were derived in Ref. 11.

In Appendix A we formulate the problem whose solution is principal for deriving the formulas of this study. Note that definition of torque in Eq. (4) is consistent with the above definition of the dipole moment which differs from the definition of the dipole moment in Refs. 2, 3, and 5. Expressions for the electric field and electric current density inside the particle, $\vec{E}_2$ and $\vec{j}_2$, read

$$\vec{E}_2 = \sum_{i=1}^3 \frac{E_i \cdot \vec{e}_i}{1 + f_{ie}} + \sum_{i=1}^3 \frac{\Pi_i(t) \vec{e}_i (f_{ie} - f_{i\sigma})}{(1 + f_{ie})(1 + f_{i\sigma})}, \quad \vec{j}_2 = \sigma_2 \vec{E}_2. \quad (5)$$

The dipole moment of the particle is determined by the following relation:

$$\vec{P} = \epsilon_0 V \sum_{i=1}^3 \frac{E_i \cdot \vec{e}_i \kappa_\epsilon}{1 + f_{ie}} + \epsilon_0 V \sum_{i=1}^3 \frac{\Pi_i(t) \vec{e}_i (\kappa_\sigma - \kappa_\epsilon)}{(1 + f_{ie})(1 + f_{i\sigma})}. \quad (6)$$

Here $f_{ie} = \kappa_\epsilon n_i$, $f_{i\sigma} = \kappa_\sigma n_i$, n_i is a depolarization factor along the i th axis, $i=1, 2, 3$ corresponds to the axes x, y, z , V is a particle volume, and ϵ_0 is permittivity of vacuum. For a spheroidal particle $n_x = n_y = (1-n)/2$, where n is depolarization factor along the axis of symmetry z . In contrast to the case considered in Refs. 10 and 11, the functions $\Pi_i(t)$ depend on time not only due to variation of the electric field with time but also due to the change of parameters $\kappa_\epsilon, \kappa_\sigma$ caused by Joule heating. In a general case when parameters of the particle ϵ_2, σ_2 vary with time, temporal evolution of functions $\Pi_i(t)$ is determined by the following equation:

$$\frac{d\Pi_i(t)}{dt} + \frac{\Pi_i(t)}{\tau_i} + \frac{1}{A_i} \frac{dA_i}{dt} \Pi_i(t) = \frac{E_i}{\tau_i}, \quad (7)$$

where $A_i = (\kappa_\sigma - \kappa_\epsilon)/(1 + f_{i\sigma})$ and $\tau_i = \tau_0(1 + f_{ie})/(1 + f_{i\sigma})$.

The system of Eqs. (2)–(6) must be supplemented with energy balance equation describing variation of spheroid temperature caused by Joule heating and equations of state which describe the dependencies of the corresponding parameters on the temperatures of the host medium, T_1 , and of the spheroid, T_2 . Hereafter we assume that the temperature of the host medium is maintained constant, so that $\kappa_\epsilon = g_\epsilon(T_2)$ and $\kappa_\sigma = g_\sigma(T_2)$, where functions g_ϵ and g_σ are the known functions of the temperature of the particle T_2 .

Neglecting spatial dispersion of temperature, the energy balance equation can be written as follows:

$$\frac{dT_2}{dt} = \frac{T_1 - T_2}{\tau_T} + Q_2, \quad (8)$$

where τ_T is a characteristic temperature relaxation time which is assumed temperature independent. The thermal energy rate released by Joule heating per unit mass is given by the following formula:

$$Q_2 = (\vec{j}_2 \cdot \vec{E}_2) \frac{1}{c\rho}, \quad (9)$$

where c is a specific heat and ρ is material density of particle. Hereafter we omit the subscript in Q_2 and T_2 , $Q_2 \equiv Q$, $T_2 \equiv T$.

The dynamics of the system is determined by Eqs. (2)–(9) which include the following parameters: characteristic time of relaxation of electric charge, τ_i [see Eq. (7)]; characteristic time of viscous rotational relaxation, $\tau_\eta = \eta_s / (\varepsilon_0 \varepsilon_1 V E_0^2)$, where E_0 is an amplitude of the electric field $E(t)$; characteristic time of the Joule heating, $\tau'_T = [(\partial Q / \partial T)_{T_S}]^{-1}$, where T_S is a characteristic temperature; and characteristic temperature relaxation time τ_T .

In this study we consider the problem which was analyzed previously in Ref. 11 without taking into account Joule heating. Hereafter we take into account temperature variation caused by Joule heating, and following Ref. 11 we adopt the following relations between the characteristic times:

$$\tau_\eta \gg \frac{1}{\omega}, \quad \tau_\eta \gg \tau_i. \quad (10)$$

We assume also that the same conditions as for the characteristic time τ_η are valid also for the characteristic times τ_T and τ'_T . Under these conditions, in the zeroth order approximation with respect to the indicated above small parameters, the solution of Eq. (7) for $E(t) = E_0 \cos \omega t$ reads

$$\Pi_x(t) = -E_0 \bar{\Pi}_x(t) \sin \theta(t), \quad \Pi_z(t) = E_0 \bar{\Pi}_z \cos \theta(t), \quad (11)$$

$$\bar{\Pi}_i(t) = \frac{\cos \omega t + \omega \tau_i \sin \omega t}{1 + \omega^2 \tau_i^2}, \quad i = x, z. \quad (12)$$

Substituting Eqs. (11) and (12) in formulas (5) and (6) and using Eqs. (3) and (9) yield formulas for $M(t)$ and $Q(t)$ which include $\theta(t)$ as the parameter. Assuming that $M(t)$ and $Q(t)$ are known we arrive at a system of two equations for $\theta(t)$ and $T(t)$. Indeed, taking into account conditions (10) for τ_η and assuming that the same conditions are valid also for τ_T and τ'_T imply that the temperature of the particle T and the angle of particle orientation do not change during the period of the electric field. These parameters vary during considerably large time intervals, $t \gg 1/\omega$. Consequently, the torque $M(t)$ and the Joule heat release rate $Q(t)$ can be averaged over the period of the electric field. In order to simplify formulas and the subsequent analysis we consider a case when $\kappa_\varepsilon \ll 1$. The latter case corresponds to a quite common situation, whereby the permittivities of the host medium and spheroid are close, $|\varepsilon_2 - \varepsilon_1| \ll 1$, while their electric conductivities are essentially different. In the limiting case $\kappa_\varepsilon \ll 1$

formula for $M(t)$, averaged over the period, reads

$$\langle M(t) \rangle = M_0 \kappa_\sigma^2 \sin 2\theta \frac{(3n-1)P(\nu, T)}{2L(\nu, T)}, \quad (13)$$

where

$$M_0 = \varepsilon_0 \varepsilon_1 V E_0^2 / 4, \quad \nu = \omega \tau_0, \quad (14)$$

$$L(\nu, T) = \{[1 + \kappa_\sigma(1-n)/2]^2 + \nu^2\}[(1 + \kappa_\sigma n)^2 + \nu^2], \quad (15)$$

$$P(\nu, T) = (1 + \kappa_\sigma(1-n)/2)(1 + \kappa_\sigma n) - \nu^2. \quad (16)$$

Formula for $Q(t)$ averaged over the period of the electric field reads

$$\langle Q(t) \rangle = Q_0 [q_S(\nu, T)/L(\nu, T) + \cos 2\theta q_N(\nu, T)/L(\nu, T)] \quad (17)$$

where

$$Q_0 = \sigma_1 E_0^2 (1 + \nu^2) / (4c\rho), \quad (18)$$

$$q_S(\nu, T) = (1 + \kappa_\sigma) \{ [1 + \kappa_\sigma(1-n)/2]^2 + (1 + \kappa_\sigma n)^2 + 2\nu^2 \}, \quad (19)$$

$$q_N(\nu, T) = \kappa_\sigma (\kappa_\sigma + 1) (2 + \kappa_\sigma(1+n)/2) (1 - 3n)/2. \quad (20)$$

Equations (18)–(20) imply that prolate and oblate spheroids are heated differently in the same range of angle. Joule heating rate in the oblate spheroid is large when orientation angle $\theta > \pi/4$, while Joule heating rate in the prolate spheroid ($n < 1/3$) is higher for $\theta < \pi/4$.

It must be noted that for spherical particle $n=1/3$ and the average torque $\langle M(t) \rangle = 0$. However, in a system with a finite electric conductivity when a condition $\kappa_\sigma < \kappa_\varepsilon$ is met and the strength of the external electric field exceeds a certain frequency dependent threshold E_c , a spherical particle loses stability and begins to rotate.¹² The latter effect does not occur in the present study because Eq. (13) is derived in the zero order approximation with respect to a parameter $\tau_i/\tau_\eta \ll 1$, where τ_i is a characteristic time of the Maxwell relaxation along x_i axis and τ_η is the defined above effective mechanical relaxation time under the influence of the external electric field. The effect of Quincke rotation is of the first order and it disappears in the zero order approximation with respect of a small parameter τ_i/τ_η .

In the case of a strongly prolate spheroid $n \rightarrow 0$,

$$\begin{aligned} \langle Q(t) \rangle &= \frac{Q_0}{2} (1 + \kappa_\sigma) \\ &\times \frac{2(1 + \nu^2) + \kappa_\sigma + \kappa_\sigma^2/4 + \cos(2\theta) \kappa_\sigma (1 + \kappa_\sigma/4)}{(1 + \nu^2)[(1 + \kappa_\sigma/2)^2 + \nu^2]} \end{aligned}$$

and

$$\langle M(t) \rangle = - \frac{M_0 \kappa_\sigma^2 \sin(2\theta) (1 + \kappa_\sigma/2 - \nu^2)}{2(1 + \nu^2)[(1 + \kappa_\sigma/2)^2 + \nu^2]}.$$

In the opposite limiting case $n \rightarrow 1$,

$$\langle Q(t) \rangle = \frac{Q_0}{2} (1 + \kappa_\sigma) \frac{2(1 + \nu^2) + 2\kappa_\sigma + \kappa_\sigma^2 - \cos(2\theta)\kappa_\sigma(2 + \kappa_\sigma)}{(1 + \nu^2)[(1 + \kappa_\sigma)^2 + \nu^2]}$$

and

$$\langle M(t) \rangle = \frac{M_0 \kappa_\sigma^2 \sin(2\theta)(1 + \kappa_\sigma - \nu^2)}{(1 + \nu^2)[(1 + \kappa_\sigma)^2 + \nu^2]}.$$

Now, let us turn to the general case. The system of Eqs. (3) and (8) can be rewritten as follows:

$$\frac{d\theta}{dt} = \frac{\kappa_\sigma^2(3n-1)P(\nu, T)}{2\tau_\eta L(\nu, T)} \sin 2\theta, \quad (21)$$

$$\frac{dT}{dt} = -\frac{T-T_1}{\tau_T} + Q_0 \left[\frac{q_S(\nu, T)}{L(\nu, T)} + \cos 2\theta \frac{q_N(\nu, T)}{L(\nu, T)} \right]. \quad (22)$$

For further analysis it is convenient to introduce the parameter T_S which is defined by the following formula:

$$\frac{T_S - T_1}{\tau_T} = Q_0 \frac{q_S(\nu, T_S)}{L(\nu, T_S)}. \quad (23)$$

The parameter T_S is equal to the temperature of the spheroid when the angle between its axis of symmetry and the direction of the external electric field is $\theta = \pi/4$. Equation (23) implies that T_S is a function of three parameters, $T_S = T_S[T_1, E^2/(c\rho), \nu]$. Since in the considered formulation of the problem these parameters remain constant, the magnitude of T_S also remains constant. Equations (21) and (22) imply that the temperature of the spheroid with orientation $\theta = \pi/4$ does not increase indefinitely for $t \rightarrow \infty$ but is equal to T_S when the following condition is satisfied:

$$\tau_T < \tau'_T, \quad \frac{1}{\tau'_T} = Q_0 \frac{d}{dT} \left[\frac{q_S(T)}{L(T)} \right]_{T=T_S}. \quad (24)$$

The latter formula for the characteristic time of the Joule heating τ'_T refines the formula for this parameter presented above. Hereafter we assume that the condition (24) is satisfied.

For the analysis of the dynamics of the system which are governed by Eqs. (21) and (22), let us introduce a new variable, $T = T_S(1+u)$. The equilibrium orientation of the spheroid is determined by the behavior of trajectory (θ, T) in the vicinity of the equilibrium states. The locations of the equilibrium states weakly depend on the behavior of the functions $L[\nu, T_S(1+u)]$, $q_N[\nu, T_S(1+u)]$, and $q_S[\nu, T_S(1+u)]$ on the argument u . The latter conclusion follows from the analysis of these functions. The character of equilibrium states is determined by the function $P[\nu, T_S(1+u)]$ since this function changes its sign while all the rest are strictly positive. The squared frequency can be written as $\nu^2 = \nu_S^2 + \Delta$, where ν_S^2 is determined by Eq. (1) for $T = T_S$,

$$\nu_S^2 = [1 + \kappa_\sigma(T_S)n][1 + \kappa_\sigma(T_S)(1-n)/2]. \quad (25)$$

Equation (16) implies that $P(\nu_S, T_S) = 0$. Now let us set $u = 0$ in functions L, q_S, q_N and assume the following formula for the electric conductivity of the spheroid as a function of temperature:

$$\sigma_2(T) = \sigma_2(T_S) + \chi_0(T - T_S), \quad (26)$$

where χ_0 is temperature independent coefficient. Then Eqs. (21) and (22) can be rewritten as follows:

$$\frac{d\mu}{d\bar{t}} = \xi_0(1-3n)(1-\mu^2)R(u, \Delta), \quad (27)$$

$$\frac{du}{d\bar{t}} = -u + \frac{1-3n}{2} s\mu,$$

where $\mu = \cos 2\theta$, $\xi_0 = \tau_T \kappa_\sigma^2 / \tau_\eta L(\nu, T_S)$, $s = \tau_T Q_0 q_N(\nu, T_S) / T_S L(\nu, T_S)$, dimensionless time $\bar{t} = t / \tau_T$, $R(u, \Delta) = \chi^2 u^2 n(1-n)/2 + \chi u[(n+1)/2 + \kappa_\sigma(T_S)n(1-n)] - \Delta$, and $\chi = \chi_0 T_S / \sigma_1$.

When $\chi_0 = 0$, i.e., temperature dependence of the electric conductivity is not taken into account, $R(u, \Delta) = -\Delta$, and Eqs. (27) describe the situation which was analyzed in Ref. 11. Equilibrium state is different for $\Delta > 0$ and $\Delta < 0$. When $\Delta < 0$ ($\nu < \nu_S$), the equilibrium state is $\mu = 1$ ($\theta = 0$) for $1-3n > 0$ and $\mu = -1$ ($\theta = \pi/2$) for $1-3n < 0$. It was noted above that the latter orientation corresponds to the case of an ideal dielectric. When the frequency of the external electric field $\nu > \nu_S$ ($\Delta > 0$), the equilibrium orientation changes to the perpendicular one.

The situation is essentially different when $\chi_0 \neq 0$. In the latter case, depending on the parameters of the problem, the number of the equilibrium orientations varies from 2 to 4. In order to elucidate this situation it is convenient to rewrite Eqs. (27) by introducing a new variable w such that $u = w s(1-3n)/2$,

$$\frac{d\mu}{d\bar{t}} = (1-3n)(1-\mu^2)\xi(w-w_L)(w-w_R), \quad (28)$$

$$\frac{dw}{d\bar{t}} = -w + \mu, \quad (29)$$

where $\xi = \xi_0 \chi^2 n(1-n)(1-3n)^2 s^2 / 8$, w_L, w_R are the roots of the quadratic equation $R(w, \Delta) = 0$, where $R(w, \Delta) = w^2 + \lambda_1 w + \lambda_2$ and λ_1, λ_2 are determined by the following formulas:

$$\lambda_1 = \frac{4n(1-n)\kappa_\sigma(T_S) + 2(1+n)}{\chi s(1-3n)(1-n)n}, \quad (30)$$

$$\lambda_2 = -\frac{8\Delta}{\chi^2 s^2 n(1-n)(1-3n)^2}.$$

Hereafter it is convenient to represent the roots w_L, w_R , where $w_L < w_R$, in the following form:

$$w_R = \frac{|\lambda_1|}{2} (-\text{sign } \lambda_1 + \sqrt{1 + \Delta_n}), \quad (31)$$

$$w_L = -\frac{|\lambda_1|}{2} (\text{sign } \lambda_1 + \sqrt{1 + \Delta_n}),$$

where the parameter $\Delta_n = 2\Delta n(1-n)/(n(1-n)\kappa_\sigma(T_S) + (1+n/2))$ is determined by the frequency Δ , depolarization coefficient n , and parameter $\kappa_\sigma(T_S)$. Using the parameter Δ_n as a second independent parameter instead of λ_2 is preferable

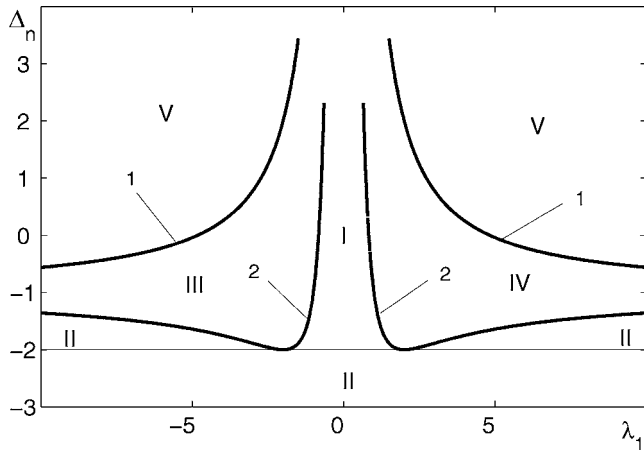


FIG. 2. Subdomains and distribution of equilibrium states: Subdomain I: four equilibrium states A_1-S , A_R-N , A_L-S , and $A_{-1}-N$; subdomain II: two equilibrium states A_1-S and $A_{-1}-N$; subdomain III: three equilibrium states A_1-N , A_L-S , and $A_{-1}-N$; subdomain IV: three equilibrium states A_1-S , A_R-N , and $A_{-1}-S$; subdomain V: two equilibrium states A_1-N and $A_{-1}-S$. Equation of curve 1: $\Delta_n = 4(1 + |\lambda_1|)/|\lambda_1|^2$. Equation of curve 2: $\Delta_n = 4(1 - |\lambda_1|)/|\lambda_1|^2$.

on physical grounds since in addition to the frequency Δ the parameter λ_2 depends on the same variables as parameter λ_1 , while for given n and $\kappa_\sigma(T_S)$ the parameters λ_1 and Δ_n are independent.

Parameters λ_1, Δ_n determine the character and the number of the equilibrium states and, consequently, determine the qualitative behavior of the system. In order to investigate this behavior, consider the system of differential equations (28) and (29). It can be easily seen that in the linear differential equation (29) the term $\mu = \cos(2\theta)$ plays a role of a “heat source” for the “temperature” w . Equation (29) implies that

$$w = w_0 \exp(-\bar{t}) + \int_0^{\bar{t}} \exp(-\tau) \mu(\bar{t} - \tau) d\tau. \quad (32)$$

Consider first the simplest case when $\xi \ll 1$. As can be seen from Eq. (28) in this case $\mu(\bar{t})$ varies much slower than $w(\bar{t})$, and for $\bar{t} \gg 1$ Eq. (32) can be rewritten as $w(\bar{t}) = \mu(\bar{t})$. Using the latter relation for eliminating $w(\bar{t})$ in Eq. (28), the solution of Eq. (28) can be found by quadratures,

$$\int_{\mu_0}^{\mu} \frac{dy}{(1-y^2)(y-w_L)(y-w_R)} = (1-3n)\xi(\bar{t}-\bar{t}_0). \quad (33)$$

Integral in Eq. (33) can be calculated via elementary functions and determines in implicit form function $\mu(\bar{t})$ for a given initial condition, $\mu_0 = \mu(\bar{t}_0)$. The values of function $\mu(\bar{t})$ depend on the sign of the parameter $(1-3n)\xi$ and the magnitudes of the roots w_L, w_R .

Under these conditions there exist five different domains which are determined by the above defined parameters λ_1, Δ_n . These domains on the plane (λ_1, Δ_n) are shown in Fig. 2. In Fig. 3 we shown phase portraits corresponding to each of these domains on the plane $(\mu, \dot{\mu})$. In order to describe a situation shown in these figures, let us denote the possible equilibrium states $\mu = (-1, w_L, w_R, 1)$ by A_i , i.e., A_{-1}, A_L, A_R, A_1 , respectively. In the following the stable equilibrium state A_i is denoted by A_i-S , and the unstable equi-

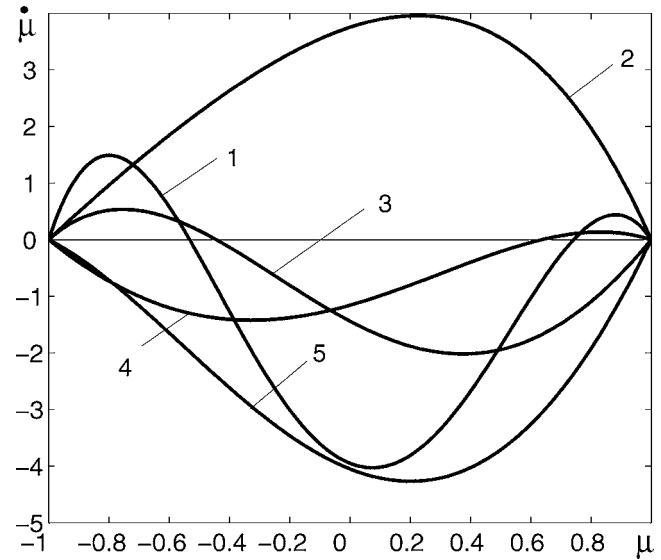


FIG. 3. Dynamic trajectories in different domains on the phase plane $(\mu, \dot{\mu})$. 1-phase portrait in the domain I ($\lambda_1=0.2, \Delta_n=35.8$); 2-phase portrait in the domain II ($\lambda_1=3.5, \Delta_n=-2.4$); 3-phase portrait in the domain III ($\lambda_1=-2.8, \Delta_n=0.73$); 4-phase portrait in the domain IV ($\lambda_1=1.03, \Delta_n=1.03$); 5-phase portrait in the domain V ($\lambda_1=-3.5, \Delta_n=2.2$); The domains I–V are shown in Fig. 2.

librium state is denoted by A_i-N . In Fig. 2 we shown the situation which occurs when condition $1-3n > 0$ is satisfied. The situation in the opposite case $1-3n < 0$ is clear from the presented below analysis.

In the subdomain I, $|\lambda_1| < 2, -1 < \Delta_n < 4(1 - |\lambda_1|)/|\lambda_1|^2$, there are four equilibrium states, A_1-S , A_R-N , A_L-S , and $A_{-1}-N$. If initially $\mu = \mu_0 > w_R$, then the system evolves in the state $\mu = 1$, while in the opposite case the system evolves to the state $\mu = w_L$. The phase portrait corresponding to this situation is shown in curve 1 in Fig. 3 where $w_L = -0.53$ and $w_R = 0.74$.

In the subdomain II, $|\lambda_1| < 2, \Delta_n < -1$ and $|\lambda_1| > 2, \Delta_n < 4(1 - |\lambda_1|)/|\lambda_1|^2$, there exist only two equilibrium states, A_1-S and $A_{-1}-N$, while the states A_L and A_R are missing because $R(w, \Delta)$ does not have real roots in this subdomain. In this subdomain spheroid evolves in the state $\theta = 0$ irrespectively of the initial position. The phase portrait corresponding to this situation is shown in curve 2 in Fig. 3.

In the subdomain III, $4(1 - |\lambda_1|)/|\lambda_1|^2 < \Delta_n < 4(1 + |\lambda_1|)/|\lambda_1|^2$, $\lambda_1 < 0$, there exist three equilibrium states, A_1-N , A_L-S , and $A_{-1}-N$, while the equilibrium state A_R is missing since $w_R > 1$. Hence the system evolves in the state with an intermediate orientation, $\mu = w_L$. The phase portrait corresponding to this situation is shown in curve 3 in Fig. 3 where $w_L = -0.42$.

In the subdomain IV, $4(1 - |\lambda_1|)/|\lambda_1|^2 < \Delta_n < 4(1 + |\lambda_1|)/|\lambda_1|^2$, $\lambda_1 > 0$, there exist three equilibrium states A_1-S , A_R-N , and $A_{-1}-S$, while the state A_L is missing since $w_L < -1$. If the initial orientation $\mu = \mu_0 > w_R = 0.64$, then the system evolves in the state $\mu = 1$, while in the opposite case the system evolves in the state $\mu = -1$. The phase portrait corresponding to this situation is shown in curve 4 in Fig. 3.

In the subdomain V as in the subdomain II, there are two equilibrium states, A_1-N and $A_{-1}-S$, while the equilibrium states A_L and A_R are missing. Here, irrespectively of the ini-

tial conditions, the system evolves in the state $\mu = -1$. The phase portrait corresponding to this situation is shown in curve 5 in Fig. 3.

The change of particle orientation in Fig. 3 is easily ascertained on the basis that in the range $\dot{\mu} > 0$ the particle moved to the right along μ -axis, while for $\dot{\mu} < 0$ the particle moves in the opposite direction. The latter criterion allows obtaining the above described pattern of particle behavior.

Consequently, temperature dependence of the parameters of the system essentially alters the behavior of the spheroid. In the case of $\chi_0 = 0$ the orientation of the spheroid is determined only by the threshold frequency of the external electric field ν_s which separates the domains with high-frequency and low-frequency orientations. When one takes into account Joule heating and temperature dependence of the electric conductivity, there appear three subdomains I, III, and IV, which separate the high-frequency orientation domain V and the low-frequency orientation domain II. The widths of these domains, and, therefore, the feasibility of observing this phenomenon, is determined by the magnitude of λ_1 . Since for large λ_1 the widths of the domains sharply decrease (see Fig. 2), small values of λ_1 are desirable for observing the effect. Formula (30) implies that in the latter case $\chi s \gg 1$. The magnitude of $Z = \chi s$ is significant in the region where $\kappa_\sigma \gg 1$. In this region $\nu^2 \sim \nu_s^2 = [1 + \kappa_\sigma(1-n)/2](1 + \kappa_\sigma n)$, $q_N = \kappa_\sigma^3(1-3n)(1+n)/4$, $q_S = \kappa_\sigma^3(1+n)^2/4$, $L = \nu_s^2 \kappa_\sigma^2(1+n)^2/4$, and $Q_0 = \sigma_1 E_0^2 (4c\rho)^{-1}$. The latter estimates imply that the parameter Z can be estimated as $Z \sim Z_0 \kappa_\sigma(1-3n)/(1+n)$, where $Z_0 = \tau_T \chi_0 E_0^2 (4c\rho)^{-1}$. Therefore, there exists a wide range of parameters where $\lambda_1 \sim 4[n(1-n)\kappa_\sigma(T_s) + (1+n)/2][Zn(1-n)]^{-1} \sim 1$.

The presented above pattern of the behavior of the system retains its validity for arbitrary magnitude of the parameter ξ . This can be proven by the linear perturbation analysis of the equilibrium points of the dynamical system (28) and (29). Indeed, since $w = \mu$ at the equilibrium states A_i , the variables w, μ in the vicinity of the equilibrium states A_i can be represented as $\mu_i = k_i + \bar{\epsilon}_\mu \exp(\gamma t)$ and $w_i = k_i + \bar{\epsilon}_w \exp(\gamma t)$. Using these formulas and performing linear perturbation analysis of Eqs. (28) and (29) we arrive at the following expressions for the growth rates:

$$\gamma_{A_\delta} = -\delta(1-3n)\xi \left(1 + \delta\lambda_1 - \frac{\lambda_1^2 \Delta_n}{4} \right), \quad \gamma_{A_\delta} = -1, \quad (34)$$

$$\gamma_{A_{L,R}} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + a_{L,R}}, \quad (35)$$

where $\delta = \pm 1$ and $a_{L,R} = \mp (1-3n)(1-w_{L,R}^2)\xi\sqrt{1+\Delta_n}|\lambda_1|/2$.

In the above classification of the equilibrium states the emphasis was on the stability of the equilibrium states. Concerning the behavior of the trajectories in the vicinity of the equilibrium states on the plane (w, μ) , it must be noted that according to Eqs. (24) and (35), all the states are stable nodes¹³ except for the equilibrium state A_L . The latter state is a stable focus¹³ on the phase plane (w, μ) provided that the following condition is satisfied:

$$[1 - |\lambda_1|^2(\Delta_n - 2\sqrt{1+\Delta_n})/4]\xi|\lambda_1|^2\sqrt{1+\Delta_n}/2 > [4(1-3n)]^{-1}.$$

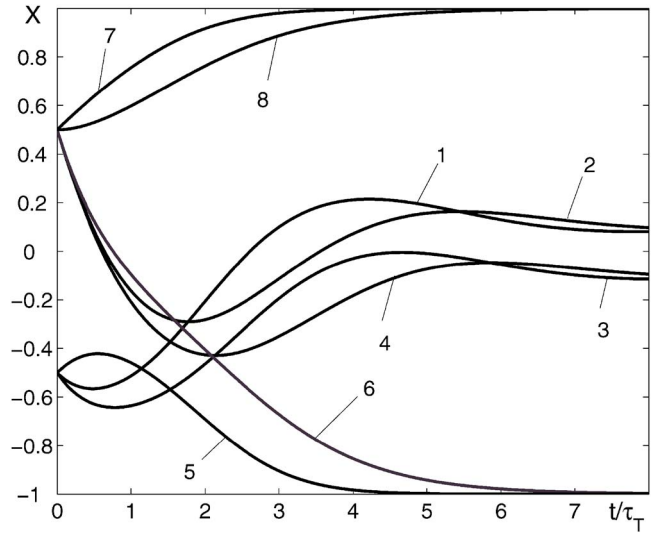


FIG. 4. (Color online) Dependencies of the dynamic variables $\mu(t)$ and $w(t)$ for various values of the parameters $p_1 = \text{sign}(\alpha)E_0^4/(E_m^2 E_c^2)$ and $p_2 = \text{sign}(\alpha)4\Delta E_m^2/E_c^2$. Curves 1 and 2: $1-\mu(t)$, $2-w(t)$, $p_1 = -1$, and $p_2 = 0.1$; curves 3 and 4: $3-\mu(t)$, $4-w(t)$, $p_1 = -1$, and $p_2 = -0.1$; curves 5 and 6: $5-\mu(t)$, $6-w(t)$, $p_1 = 1$, and $p_2 = 0.1$; curves 7 and 8: $7-\mu(t)$, $8-w(t)$, $p_1 = 1$, and $p_2 = 0.1$.

Up until now it was assumed that the parameters of the system are arbitrary. Large number of governing parameters precludes from investigating the system in the whole range of the parameters. Let us consider now a simple but readily physically realizable particular case when a spheroid is a thin needle with sufficiently large electric conductivity so that the following conditions are met:

$$\kappa_\sigma \gg 1, \quad n \ll 1, \quad \kappa_\sigma n \ll 1. \quad (36)$$

Under these conditions the coefficients in Eqs. (28) and (29) can be substantially simplified and Eq. (28) can be rewritten as follows:

$$\frac{d\mu}{dt} = \text{sign}(\chi_0) \frac{E_0^4}{E_m^2 E_c^2} (1 - \mu^2) \left[w - \text{sign}(\chi_0) \frac{4E_m^2 \Delta}{E_0^2 \kappa_\sigma} \right], \quad (37)$$

where E_m and E_c are the characteristic values of the electric fields, $E_m^2 = 4c\rho(\tau_T|\chi_0|)^{-1}$ and $E_c^2 = \eta(8\tau_T\epsilon_0\epsilon_1)^{-1}$, and $\eta = \eta_s/V$ is the dynamic viscosity of the host fluid. The coefficient χ_0 in Eq. (37) is related with temperature coefficient of resistivity α by the formula $\chi_0 = -\alpha\sigma_2$, and χ_0 can be either positive or negative. For estimating the magnitude of E_m assume that $\alpha \approx 10^{-3} \text{ K}^{-1}$, $c\rho \approx 10^6 \text{ kJ kg}^{-1} \text{ K}^{-1}$, and electric conductivity $\sigma_2 \approx 1 \text{ S m}^{-1}$. These values of the parameters imply that $E_m \approx 10^5 \text{ V/m}$. The amplitude of the electric field E_c can be compared to E_c^Q , where E_c^Q is the magnitude of the electric field, whereby the Quincke effect can be observed in the fluid if the condition $\sigma_2 < \sigma_1$ is satisfied. This estimate is given by the following formula:

$$E_c^Q \approx \sqrt{\tau_0/\tau_T}.$$

Numerical solutions of Eqs. (28) and (37) are shown in Fig. 4. Taking into account that the functions $\mu(t)$ and $w(t)$ vary in the same range, we plotted these functions in the same figure. In this figure we denoted the numerically calculated plots of these functions by $X(t)$, whereby the curves

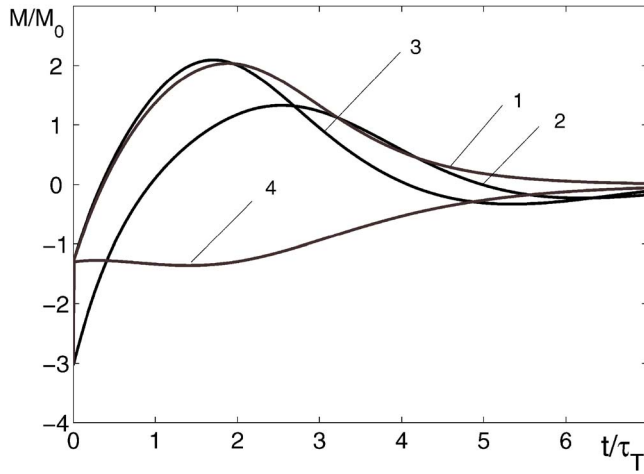


FIG. 5. (Color online) Dependencies of the torque $M(t)/M_0$ for various values of the parameters $p_1 = \kappa_\sigma \text{sign}(\alpha) E_0^4 / (E_m^2 E_c^2)$ and $p_2 = \text{sign}(\alpha) 4 \Delta E_m^2 / E_0^2$. Curve 1: $p_1 = -5$ and $p_2 = 0.2$; curve 2: $p_1 = -5$ and $p_2 = -0.2$; curve 3: $p_1 = 5$ and $p_2 = -0.2$; curve 4: $p_1 = 5$ and $p_2 = 0.2$.

with odd numbers correspond to the function $\mu(t)$, while the curves with even numbers correspond to the function $w(t)$. Inspection of Fig. 4 reveals that the behavior of these functions complies with the presented above analysis. As has been noted above in the domain III in Fig. 2 depending on the strength of the external electric field E_0 , the equilibrium point can be either a node or a focus. In the latter case the orientation of the needle experiences damping oscillations. Such behavior of the system can be seen in curves 2–4 in Fig. 4. Time dependences of temperature and torque, $T(t)$ and $M(t)$, are determined by the following formulas:

$$T(t) = T_S \left[1 - \frac{E_0^2}{E_m^2 \alpha T_S} w(t) \right], \quad (38)$$

$$M(t) = -M_0 \text{sign}(\chi_0) \frac{E_0^2}{E_m^2} \kappa_\sigma \sqrt{1 - \mu^2(t)} \times \left[w(t) - \text{sign}(\chi_0) \frac{4 \Delta E_m^2}{E_0^2} \right].$$

Formulas (38) imply that the time dependence of normalized temperature $T/T_S - 1$ is the same [with the accuracy of the $\text{sign}(\chi_0)$] as $w(t)$, and it is shown in Fig. 4. The time dependencies of the normalized torque $M(t)/M_0$ and the orientation angle $\theta(t) = \cos^{-1}[\mu(t)]/2$ are shown in Figs. 5 and 6.

Hence taking into account temperature dependence of the conductivity of a particle embedded into a host fluid qualitatively changes the dynamics of the system. The above presented analysis implies that depending on the sign of temperature coefficient of the resistivity α , the orientation of the embedded particle and the orientation relaxation behavior can be changed by varying the amplitude and frequency of the applied external electric field.

III. TUMBLING OF THE AXIS OF THE SPHEROID IN ANISOTROPIC TEMPERATURE FIELD

Another effect associated with temperature dependence of the parameters of the problem is tumbling of the axis of

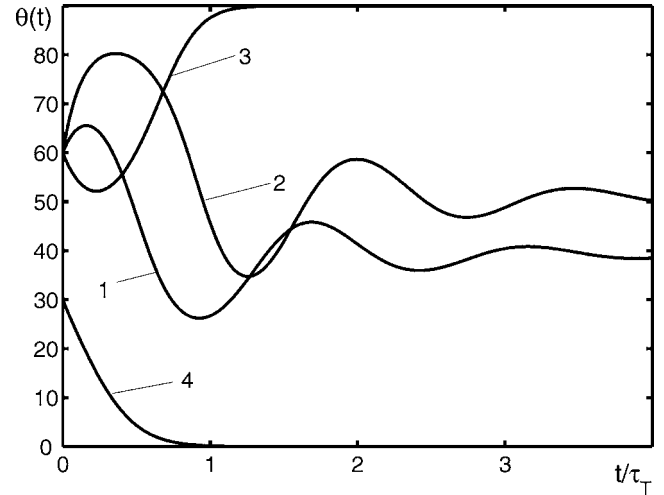


FIG. 6. Dependencies of the orientation angle $\theta(t)$ for various values of the parameters $p_1 = \text{sign}(\alpha) E_0^4 / (E_m^2 E_c^2)$ and $p_2 = \text{sign}(\alpha) 4 \Delta E_m^2 / E_0^2$. Curve 1: $p_1 = -5$ and $p_2 = 0.2$; curve 2: $p_1 = -5$ and $p_2 = -0.2$; curve 3: $p_1 = 5$ and $p_2 = -0.2$; curve 4: $p_1 = 5$ and $p_2 = 0.2$.

the spheroid in the anisotropic temperature field. Leaving the detailed analysis of this problem for a separate study, hereafter we demonstrate the feasibility of such phenomenon.

Consider the case when a temperature of the host medium in the vicinity of the spheroid $T_1(\theta)$ depends on the angle θ measured relatively to the direction of the external electric field and assume that $\partial T_1(\theta) / \partial \theta \gg Q \tau_T$ [see Eqs. (9) and (22)]. Under these assumptions Eq. (21) remains unchanged, while Eq. (22) is replaced by

$$\frac{dT}{dt} = - \frac{T - T_1(\theta)}{\tau_T}. \quad (39)$$

Assume that the frequency of the external electric field satisfies the equation $P(\nu, T_0) = 0$, where $T_0 = T_1(\theta_0)$. Temperature $T_1(\theta)$ in the vicinity of θ_0 , $T_1(\theta) = T_1(\theta_0) + (\partial T_1 / \partial \theta)_{\theta_0} (\theta - \theta_0)$, and Eq. (39) can be rewritten as follows:

$$\frac{d\Gamma}{dt} + \Gamma = \alpha(\theta - \theta_0), \quad (40)$$

where $\Gamma = T - T_0$ and $\alpha = (\partial T_1 / \partial \theta)_{\theta_0}$. Taking into account that $P(\nu, T_0) = 0$, Eq. (21) in the vicinity of θ_0 can be rewritten as

$$\frac{d\theta}{dt} = \beta \Gamma, \quad (41)$$

where $\beta = \tau_T \kappa_\sigma^2 (3n - 1) [\partial P(\nu, T_0) / \partial T]_{T_0} / 2 \tau_\eta L(\nu, T_0) \sin 2\theta_0$. Equations (40) and (41) imply the following equation:

$$\frac{d^2 \Gamma}{dt^2} + \frac{d\Gamma}{dt} - \alpha \beta \Gamma = 0. \quad (42)$$

In the case $\alpha \beta < 0$, the axis of the spheroid performs damping oscillations around the direction θ_0 [(θ_0, T_0) is a focus]. In the opposite case $\alpha \beta > 0$, the state (θ_0, T_0) is unstable.

IV. CONCLUSIONS

We investigated the dynamics of orientation of spheroid immersed into a host medium under the action of the alter-

nating external electric field when a particle and a host medium have finite electric conductivities and parameters of the system depend on temperature. It was shown that the rate of Joule heating of the particle depends on the orientation of its axis of symmetry with respect to the direction of the electric field. If the electric conductivity of particle strongly varies with temperature, the Joule heating changes particle orientation dynamics and results in appearance of the new equilibrium orientations. We studied the effect of the frequency of the electric field on the dynamics of particle rotation and on the direction of the equilibrium orientation, taking into account Joule heating. It was demonstrated that under certain conditions in the anisotropic temperature field the axis of particle can perform damping oscillations. It must be noted that taking into account the particle inertia does not change qualitatively the particle behavior. It is demonstrated that the effect considered in the present study can be realized for the amplitude of the external electric field which is smaller than the amplitude of the electric field required for Quincke rotation.

APPENDIX A: SYSTEM OF GOVERNING EQUATIONS

The results derived in this study are based on the system of equations which includes Poisson equation (A1), equation of conservation of electric charge (A2), and equations which relate the principal electrodynamic variables in the problem (A3),

$$\vec{\nabla} \cdot \vec{D} = \rho_{\text{ex}}, \quad (\text{A1})$$

$$\frac{\partial \rho_{\text{ex}}}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0, \quad (\text{A2})$$

$$\vec{D} = \varepsilon_0 \vec{E}, \quad \vec{j} = \vec{j}_\sigma + \vec{j}_c, \quad \vec{j}_\sigma = \sigma \vec{E}, \quad \vec{E} = -\vec{\nabla} \varphi, \quad (\text{A3})$$

where $\vec{j}_c$ is a convective electric current caused by a macroscopic motion of the charged particle.

The host medium with an embedded particle can be considered as a piecewise homogeneous medium. Since a charge is localized at the inhomogeneous inclusions, in the case of a piecewise homogeneous medium, it accumulates at the interface boundaries. The density of a free surface charge γ is determined by the following relations:

$$\int \rho_{\text{ex}} dV = \int \gamma dS$$

or

$$\rho_{\text{ex}} = \gamma \delta(u) |\vec{\nabla} u|,$$

where $\delta(u)$ is Dirac's delta function, $u = F(x, y, z)$, and $u=0$ is an equation of the particle surface. Equations (A1)–(A3) yield the following boundary conditions at the particle-host medium interface:

$$[\vec{N} \cdot \vec{D}] = \gamma, \quad [\vec{N} \cdot \vec{j}_\sigma] = -\frac{\partial \gamma}{\partial t} - \vec{v}_s \cdot [\vec{\nabla} - \vec{N}(\vec{N} \cdot \vec{\nabla})] \gamma, \quad (\text{A4})$$

where $[A] = A_+ - A_-$, A_+ , and A_- are the values of the function A at the external and internal surfaces, respectively, $\vec{v}_s$ is the velocity at the surface of the particle, and $\vec{N}$ is the external unit normal vector at the particle surface.

A system of equations (A1) and (A2), relations (A3), and boundary conditions (A4) provide the mathematical formulation of the problem that is considered in this study. The solution of the above problem is determined in a system of coordinates associated with the rotating spheroid. Electric potential φ is convenient to represent as $\varphi = \varphi_\varepsilon + \varphi_\sigma$. Here φ_ε is a potential caused by a local polarization occurring at the microscopic time scale, and it is equal to the total potential in the case of the ideal dielectric. The additional potential φ_σ arises due to the finite electric conductivity, and it is caused by the accumulation of the electric charge at the surface of the particle. Each of the equations (A1) and (A2) together with the corresponding boundary conditions (A4) determines the same potential φ . Finally, equating the potential found by solving Eq. (A1) to the potential determined by solving Eq. (A2) yields Eq. (7) in this study (for more details see Refs. 10 and 11).

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