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Dynamics of electromagnetic field in imploding conducting shell

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In this study we derived explicit analytical solutions of two problems. First is a problem of accumulating of magnetic fields in imploding shells with different geometries. The second problem is determining the eigenmodes of a resonator with moving walls. Using the obtained exact solutions we found the regions of validity of an adiabatic approximation when the frequency of the resonator follows the instantaneous values of geometrical parameters, and the amplitude is determined by a conservation of a magnetic flux. The obtained solutions are of interest in design of devices for generation of ultrastrong magnetic fields and powerful electromagnetic radiation. © *1998 American Institute of Physics.* [S0021-8979(98)01011-1]

I. INTRODUCTION

One of the efficient methods to transform magnetic and electric fields into electromagnetic beams of high power density is a rapid change of the geometry of an accumulating system with initial nonzero electromagnetic field. Typical examples of such systems are magnetoimplosive generators (see, e.g., Refs. 1–5) and z-pinch plasma which has been investigated theoretically and experimentally for many years. A characteristic feature of such systems is the existence of the initial electromagnetic field in a cavity surrounded by a conducting medium. When for some reasons a volume of the cavity rapidly decreases, and its walls move fast enough, the amplitude of the electromagnetic field increases due to concentration of the electromagnetic field.

If the cavity has ideally conducting walls the magnitude of the average magnetic field can be found from conservation of a magnetic flux:

$$\frac{\partial\tilde{\Phi}}{\partial t}=0. \tag{1}$$

Here $\tilde{\Phi}=\int \mathbf{H}d\mathbf{S}$ and $\mathbf{H}$ is a magnetic field inside a cavity and integration is performed over a cross section of the cavity. According to Faraday's law $\varepsilon=-(1/c)(\partial\tilde{\Phi}/\partial t)$, where ε is an electromotive force, condition (1) provides a finite magnitude of electric current in a conductor with an infinite conductivity. Now taking into account that

$$\frac{\partial\tilde{\Phi}}{\partial t}=\int\frac{\partial\mathbf{H}}{\partial t}\,d\mathbf{S}+\oint\mathbf{H}\cdot[\mathbf{v}\times\boldsymbol{\tau}(\ell)]d\ell,$$

where $\boldsymbol{\tau}(\ell)$ a unit vector tangent to the closed contour $\mathbf{r}(\ell)(\boldsymbol{\tau}(\ell)=\partial\mathbf{r}/\partial\ell)$ and $\mathbf{v}$ is contour's velocity at $\mathbf{r}(\ell)$. The latter equation and Maxwell equation in an integral form

$$-c\oint(\mathbf{E}\cdot\boldsymbol{\tau})d\ell=\int\frac{\partial\mathbf{H}}{\partial t}\,d\mathbf{S},$$

where $\mathbf{E}$ is electric field, together with Eq. (1) yield one of the boundary conditions at the conductor's surface,

$$\boldsymbol{\tau}\cdot\left(\mathbf{E}+\frac{1}{c}\,\mathbf{v}\times\mathbf{H}\right)=0. \tag{2}$$

Condition (2) provides that $\varepsilon=0$ not only when Faraday's law is valid but in a more general case $\varepsilon=f(v/c)\partial\tilde{\Phi}/\partial t$ where f is an arbitrary finite function of parameter v/c. Therefore it is conceivable to suggest that when velocity of a shell is normal to its surface and $v=|\mathbf{v}|$ is the same at all locations at the surface, condition (2) is valid without relativistic restrictions. In this study we will consider only such cases when condition (2) is valid exactly for the ideal conductor.

In various practical problems it is necessary to determine not only the volume-averaged value of a magnetic field but also spatial distribution of electromagnetic field inside cavity which can be found by solving Maxwell equations.

Another problem which is of fundamental and practical importance, is to determine frequency characteristics of the resonator with moving walls. While a case of the accumulation of a magnetic field was investigated in Ref. 6, to the best of our knowledge variation of eigenmodes of a resonator with moving walls was not considered before.

In order to solve these problems the boundary conditions (2) must be supplemented with boundary conditions for the normal components of magnetic (H_n) and electric field (E_n):

$$E_n=0,\quad H_n^-=H_n^+, \tag{3}$$

where H_n^- and H_n^+ are the values of a magnetic field at the internal and external sides of the shell's surface, respectively. The first of the conditions (3) accounts for the absence of the static charges in the system and for the absence of the normal component of the electric field inside the conductor. The second in the conditions (3) is the consequence of $\boldsymbol{\nabla}\cdot\mathbf{H}=0$.

Hereafter a geometry of the electromagnetic field is described by vector potential $\mathbf{A}$ which allows one to determine electric and magnetic fields (see, e.g., Ref. 7):

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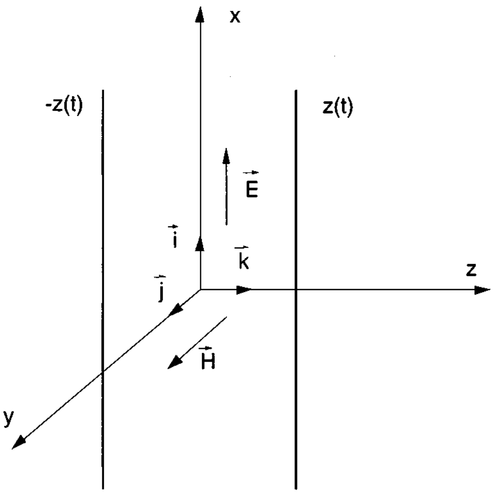

FIG. 1. Coordinate system.

$$\mathbf{E}=-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{H}=\nabla\times\mathbf{A}. \tag{4}$$

Vector potential itself is determined by the following equation which can be derived from the Maxwell equations for the fields $\mathbf{E}$ and $\mathbf{H}$ with an additional condition $\nabla\cdot\mathbf{A}=0$:

$$\nabla^2\mathbf{A}=\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}. \tag{5}$$

Equation (5) with boundary conditions (2) and (3) determines a full set of waves which are eigenmodes of a resonator and, therefore, they determine spatial and time dependence of the fields excited in a resonator.

In this study we analyze redistribution of an electromagnetic field in a one-dimensional plane resonator and a spherical resonator with the walls moving at constant velocity. We investigate also accumulating of magnetic field in plane, cylindrical, and spherical symmetry.

II. EIGENMODES OF PLANE ONE-DIMENSIONAL RESONATOR WITH WALLS MOVING WITH A CONSTANT VELOCITY

Consider two ideally conducting parallel plates moving with a constant velocity v in the opposite directions of z axis (see Fig. 1). Locations of the left and right plates, $z_-(t)$ and $z_+(t)$ is given by the following formula:

$$z_\pm(t)=\pm(z_0+vt),$$

where z_0 is an initial location of the right plate at $t=0$.

We will consider one-dimensional plane wave with a wave vector $\mathbf{k}$ normal to the walls of a resonator. In this case vector potential can be chosen as follows:

$$\mathbf{A}=[A_x(z,t),0,0]. \tag{6}$$

General solution of Eq. (5) can be represented as a linear combination of arbitrary functions $F(t\pm z/c)$ (see, e.g., Ref. 7). Then a component of a vector potential $A_x(z,t)$ can be written as

$$A_x(z,t)=\sum_{\sigma=\pm 1} C_\sigma A_\sigma\left[\frac{z_0+v(t+\sigma z/c)}{z_0}\right], \tag{7}$$

where C_σ are numerical coefficients. A structure of an argument in functions $A_\pm\{[z_0+v(t\pm z/c)]/z_0\}$ ensures that they satisfy Eq. (5). In order to simplify notations introduce dimensionless variables $\tau=ct/z_0$ and $\overline{z}=z/z_0$.

Using an identity $z_0+v(t+\sigma z/c)\equiv z_0(1+\beta\tau)(1+\sigma\beta\kappa)$, where $\beta=v/c$ and $\kappa=\overline{z}/(1+\beta\tau)$, we will look for the solution of a boundary value problem (2) in the form of a power function:

$$A_\sigma^\lambda=a_0(1+\beta\tau)^\lambda(1+\sigma\beta\kappa)^\lambda, \tag{8}$$

where $\lambda(\beta)$ is a complex function and a_0 is a constant. Spectrum of the admissible values $\lambda(\beta)$ is determined by the spatial parity of a corresponding solution. Consider a case when

$$H(z,t)=H(-z,t). \tag{9}$$

In this case vector potential $A_x(z,t)$ satisfies the condition $A_x(z,t)=-A_x(-z,t)$ and it can be written as

$$A_x(z,t)=a_0(1+\beta\tau)^\lambda\sum_{\sigma=\pm 1}\sigma(1+\sigma\beta\kappa)^\lambda. \tag{10}$$

Then according to Eq. (4) formulas for electric and magnetic fields read:

$$E_x(z,t)=-\frac{a_0}{z_0}\lambda\beta(1+\beta\tau)^{\lambda-1}\sum_{\sigma=\pm 1}\sigma(1+\sigma\beta\kappa)^{\lambda-1}, \tag{11}$$

$$H_y(z,t)=\frac{a_0}{z_0}\lambda\beta(1+\beta\tau)^{\lambda-1}\sum_{\sigma=\pm 1}(1+\sigma\beta\kappa)^{\lambda-1}. \tag{12}$$

Due to the symmetry of the problem if a condition (2) is satisfied at one of the plates it is also satisfied at another plate. At a plate $\overline{z(t)}=z_0+vt$, Eqs. (2), (11), and (12) yield:

$$E_x-\beta H_y|_{\kappa=1}=0,$$

or

$$(1+\beta)^\lambda=(1-\beta)^\lambda. \tag{13}$$

Therefore,

$$\lambda=\frac{2\pi k}{\ell n\left(\frac{1+\beta}{1-\beta}\right)}i, \quad k=0,1,\ldots . \tag{14}$$

Formula (14) provides a set of eigenvalues $\lambda_k(\beta)$ when condition (9) is satisfied. For an odd spatial parity of a magnetic field $H(z,t)=-H(-z,t)$,

$$\lambda=\frac{2\pi(k+1)}{\ell n\left(\frac{1+\beta}{1-\beta}\right)}i, \quad k=0,1,\ldots .$$

Equations (11)–(14) determine eigenmodes of the electromagnetic field inside a cavity between two parallel plates moving in the opposite directions with a constant velocity.

In a dimensional form using Eq. (14), Eqs. (11) and (12) can be rewritten as follows:

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$$E_x=-\sum_{\sigma=\pm 1}\frac{\sigma a_0}{z_0+vt+\sigma\beta z}\times\exp\left[i\frac{\pi k}{\beta f(\beta)}\ell n\left(1+\frac{vt}{z_0}+\frac{\sigma\beta z}{z_0}\right)\right], \quad (15)$$

$$H_y=\sum_{\sigma\pm 1}\frac{a_0}{z_0+vt+\sigma\beta z}\times\exp\left[i\frac{\pi k}{\beta f(\beta)}\ell n\left(1+\frac{vt}{z_0}+\frac{\sigma\beta z}{z_0}\right)\right], \quad (16)$$

where

$$f(\beta)=\frac{1}{2\beta}\ell n\left(\frac{1+\beta}{1-\beta}\right)=\sum_{n=0}^{\infty}\frac{\beta^n}{n+1}\frac{[1+(-1)^n]}{2}, \quad \beta<1. \quad (17)$$

Equations (15) and (16) are valid in the range $-(z_0+vt)<z<z_0+vt$ and $\beta<1$. When $vt\ll z_0$ and $\beta\ll 1$, keeping only the first nonvanishing terms in the expansion over parameters $\beta\ll 1$ and vt/z_0 we find that

$$E_x=-\frac{2a_0}{z_0}i\exp\left(i\frac{\pi kct}{z_0}\right)\sin\left(\frac{\pi kz}{z_0}\right), \quad (18)$$

$$H_y=\frac{2a_0}{z_0}i\exp\left(i\frac{\pi kct}{z_0}\right)\cos\left(\frac{\pi kz}{z_0}\right). \quad (19)$$

Equations (18) and (19) describe eigenmodes of a one-dimensional plane resonator with a fixed position of the walls $z=z_0$.

Since the walls move with a constant velocity the initial time can be selected arbitrary, and Eqs. (18) and (19) can be rewritten using a substitution $t\to t-t_1$ and $z_0\to z(t_1)$. Then it can be easily seen that when $\beta\ll 1$ the system is adiabatic, e.g., electromagnetic field frequency varies adiabatically in accordance with variation of a distance between the walls, e.g., $\omega_k(t)=\pi kc/z(t)$, and the amplitude of the electromagnetic field varies in accordance with Eq. (1) of the conservation of a magnetic flux.

In the next approximation in the parameters vt/z_0 and β in addition to the Doppler frequency correction which is associated with the substitution $k\to k/f(\beta)\approx k(1-\beta^2/3)$, there begins formation of a wave packet, i.e., a mode loses monochromatism.

Equation (16) also describes a case of an accumulation of a magnetic field by imploding wall which was considered in Ref. 6. Thus performing a substitution of $v\to -v$ in Eq. (16) and setting $k=0$ recovers Eq. (2.9) from Ref. 6:

$$H_y=\frac{2a_0}{v}\frac{T-t}{(T-t)^2-\frac{z^2}{c^2}}, \quad \frac{|z|}{c}<T-t,$$

where $T=z_0/v$ is a moment of the collapse of the cavity.

In the above analysis we assumed that the walls of the resonator are ideal conductors. The case with a finite conductivity must be considered separately and is not investigated in this study. Another problem which considerably complicates a solution is that the interface between a conducting and a dielectric media can move with a velocity different from a fluid velocity, and under certain conditions there are no hydrodynamic flows. Thus, e.g., in a case of a phase transition from metal to dielectric there are no hydrodynamic flows if a phase transition is not accompanied by a change of a density. In a general case v in a boundary condition (2) is a material velocity while the boundary condition is formulated at the surface separating dielectric and conducting media. Therefore, without hydrodynamic flow the boundary condition (2) reduces to $\mathbf{E}=0$. Equation (13) can be rewritten using Eq. (11) as $(1+\beta)^{\lambda-1}=(1-\beta)^{\lambda-1}$, and instead of Eq. (14) we have $\lambda=1+2\pi ki/\ell n[(1+\beta)/(1-\beta)]$. Substituting the determined value of λ into Eqs. (11) and (12) it can be seen that without a hydrodynamic flow only phases of the eigenmodes change while a cumulative change of the amplitude does not occur. In a case when $k=0$, i.e., a pure magnetostatic case, magnetic field does not change contrary to the case with a hydrodynamic flow. Thus the wave of an electric conductivity alone does not cause an accumulation of a magnetic field. Consider now a case of a spherical resonator.

III. DYNAMICS OF ELECTROMAGNETIC FIELD IN A SPHERICAL RESONATOR WITH AN EXTERNAL WALL MOVING WITH A CONSTANT VELOCITY

Consider a spherical resonator with a wall moving with a constant radial velocity $\mathbf{v}=v\mathbf{n}$ ($v=$const), where $\mathbf{n}$ is a unit vector directed in a radial direction from a center of a sphere. Assume that electromagnetic field in a resonator has azimuthal symmetry and is determined by a toroidal vector potential $\mathbf{A}=A_\varphi\mathbf{e}_\varphi$ and magnetic field $\mathbf{H}=H_\theta\mathbf{e}_\theta+H_n\mathbf{n}$ is determined by the following formulas:

$$H_\theta=-\left(\frac{\partial A_\varphi}{\partial r}+\frac{A_\varphi}{r}\right),\qquad H_n=\frac{ctg(\theta)}{r}A_\varphi+\frac{1}{r}\frac{\partial A_\varphi}{\partial\theta}, \quad (20)$$

where

$$\begin{pmatrix}\mathbf{n}\\ \mathbf{e}_\theta\\ \mathbf{e}_\varphi\end{pmatrix}=\begin{pmatrix}\sin\theta\cos\varphi & \sin\vartheta\sin\varphi & \cos\theta\\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta\\ -\sin\varphi & \cos\varphi & 0\end{pmatrix}\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{pmatrix}.$$

Solution of Eq. (5) in this case can be represented as follows:

$$A_\varphi=A_\ell(r,t)P_\ell^1(\cos\theta), \quad \ell=1,2,...,$$

where P_ℓ^1 are associated Legendre polynomials, and functions $A_\ell(r,t)$ are determined by the following equation:

$$\frac{1}{c^2}\frac{\partial^2 A_\ell}{\partial t^2}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\ell}{\partial r}\right)-\frac{\ell(\ell+1)}{r^2}A_\ell. \quad (21)$$

Any solution of Eq. (21) without singularity at $r=0$ can be represented in the following form (see Appendix A):

$$A_\ell(r,t)=r^\ell\left(\frac{\partial}{r\partial r}\right)^\ell\left[\frac{f(t+r/c)-f(t-r/c)}{r}\right], \quad (22)$$

where $f(x)$ is an arbitrary function.

Let us select function f in the following form:

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$$f(t)=\left(\frac{r_0+vt}{r_0}\right)^{\lambda},\tag{23}$$

where r_0 is a radius of a sphere at $t=0$.

According to Eqs. (22) and (23) $A_\ell(r,t)$ can be represented as

$$A_\ell(r,t)=a_0\left[\frac{\overline{r(t)}}{r_0}\right]^{\lambda-\ell-1}\Phi_\ell\left[\frac{r}{\overline{r(t)}}\right],\quad \overline{r(t)}=r_0+vt,\tag{24}$$

where functions $\Phi_\ell(\chi)$ are determined by the following formula:

$$\Phi_\ell(\chi)=\chi^\ell\hat{P}_\ell\sum_{\sigma=\pm1}\frac{\sigma(1+\sigma\beta\chi)^{\lambda}}{\chi},\quad \chi=\frac{r}{\overline{r(t)}},\tag{25}$$

where an operator $\hat{P}_\ell$ is defined as

$$\hat{P}_\ell=\left(\frac{1}{\chi}\frac{d}{d\chi}\right)^{\ell}.$$

Using Eqs. (2) and (4) and taking into account a spherical symmetry of the problem, condition (2) can be rewritten as

$$\frac{\partial A_\ell}{\partial t}+v\left(\frac{\partial A_\ell}{\partial r}+\frac{A_\ell}{r}\right)_{r=\overline{r(t)}}=0.$$

Thus using Eqs. (24) and (25) we find that

$$(\lambda-\ell-1)\Phi_\ell(1)=0.\tag{26}$$

Assuming that $\lambda\neq\ell+1$ Eq. (26) yields

$$\Phi_\ell(1)=0,\tag{27}$$

Eqs. (24), (25), and (27) provide a full solution of the problem.

Consider magnetic dipole modes of a spherical resonator (see Ref. 8, Chap. 10, Sec. 90), i.e., modes with $\ell=1$. Hereafter it is convenient to represent $\lambda(\beta)$ as

$$\lambda(\beta)=1+i\,\frac{\alpha(\beta)}{\beta}.\tag{28}$$

Using Eq. (25) an expression for a function $\Phi_1(\chi)$ can be written as follows:

$$\Phi_1(\chi)=\sum_{\sigma=\pm1}\left(\frac{i\alpha}{\chi}-\frac{\sigma}{\chi^2}\right)\exp\left[i\,\frac{\alpha}{\beta}\,\ell n(1+\sigma\beta\chi)\right].\tag{29}$$

Equation (27) yields the following transcendental equation:

$$\tan[\alpha(\beta)f(\beta)]=\alpha(\beta),\tag{30}$$

where function $f(\beta)$ is determined by Eq. (17).

Function $\alpha(\beta)$ can be expanded into power series of parameter β^2

$$\alpha(\beta)=\alpha_0+\sum_{n=1}^{\infty}c_n\beta^{2n},\tag{31}$$

which converge in the range $\beta<1$. Parameter α_0 is a root of the following transcendental equation (see, e.g., Ref. 8);

$$\tan\,\alpha_0=\alpha_0.\tag{32}$$

Equation (32) has an infinite number of roots $|\alpha_0^i|>\pi$ which correspond to different functions $\alpha^i(\beta)$ such that $\alpha^i(\beta)\to\alpha_0^i$ when $\beta\to0$. Using Eqs. (30)–(32) allows one to determine correction of any order in parameter β^2. Thus, e.g., for the first correction we find that $c_1=-1/3\alpha_0$ and $\alpha=\alpha_0-\beta^2/3\alpha_0$. The minimum value of α_0^i is $\alpha_0=4.49$. In the main order from Eq. (29) we find that

$$\Phi_1(\chi)\approx\frac{2i\alpha_0}{\chi}\cos(\alpha_0\chi)-\frac{2i}{\chi^2}\sin(\alpha_0\chi).$$

Note also that $\Phi_1(\chi)=-2i\alpha_0^2j_1(\alpha_0\chi)$, where $j_1(x)$ are spherical Bessel functions (see, e.g., Ref. 9).

The above analysis corresponds to a case when at $\beta\to0$ the obtained solutions coincide with solutions of Eq. (5). In this case a total magnetic flux $\tilde{\Phi}=0$. A different physical situation occurs in a case of cumulation of an initial nonzero magnetic field. In the latter case $\tilde{\Phi}=\text{const}\neq0$ and at $\beta\to0$ a solution $A(r,t)$ coincides with the solution of a Laplace equation

$$\nabla^2A=0.\tag{33}$$

Therefore without motion of a boundary the electric field is zero and, as it will be shown further, it is of the order of β^2. A formal difference between these two cases is that in the latter case a coefficient $\lambda(\beta)$ does not have singularity, and solution of Eq. (5) can be represented as power series in parameter β^2. Functions $\Phi_\ell(\chi)$ in this case do not satisfy a condition $\Phi_\ell(1)=0$, and condition (26) can be satisfied only if $\lambda=\ell+1$. Therefore a solution in this case is given by Eqs. (24) and (25) with $\lambda=\ell+1$.

In order to demonstrate the validity of the latter statement it is convenient to represent expression (25) through a hypergeometric function. Indeed, if we present a solution in the form (24), function $\Phi_\ell(\chi)$ satisfies the following equation:

$$\left[\frac{\partial^2}{\partial\chi^2}+\frac{2}{\chi}\frac{\partial}{\partial\chi}-\frac{\ell(\ell+1)}{\chi^2}\right]\Phi_\ell(\chi)=\beta^2\hat{S}\Phi_\ell(\chi),\tag{34}$$

where operator $\hat{S}=\chi^2\partial^2/\partial\chi^2+2(1-\gamma)\chi\partial/\partial\chi+\gamma(\gamma-1)$ and $\gamma=\lambda-\ell-1$.

The solution of Eq. (34) can be written as

$$\Phi_\ell(\chi)=\chi^\ell F\left(\frac{s}{2},\frac{s}{2}-\frac{1}{2};\ell+\frac{3}{2};\beta^2\chi^2\right),\tag{35}$$

where $F(a,b;c;z)$ is a hypergeometric function (see, e.g., Ref. 9) and $s=2(\ell+1)-\lambda$.

Assuming that λ does not depend upon parameter β and expanding $\Phi_\ell(\chi)$ into Taylor series in parameter β^2, we find that $\Phi_\ell(\chi)$ cannot satisfy a condition $\Phi_\ell(1)=0$, e.g., according to Eq. (26) $\lambda=\ell+1$.

Since both functions (25) and (35) are solutions of the same equation at $\chi\to0$ and they are both $\propto\chi^\ell$, they can differ only by a constant coefficient $m=m(\ell,\lambda,\beta)$ which can be determined by equating the fist term in power series expansion in parameter $(\beta\chi)^2$. Thus using Eq. (25) allows one to derive a representation for a hypergeometric function under certain relations between its parameters. We do not present here formulas for normal and tangential components

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of magnetic field which can be easily derived from Eqs. (20), (24), (25), or (35). Now consider a case with a cylindrical geometry.

IV. ACCUMULATION OF A MAGNETIC FIELD IN A CYLINDRICAL SHELL

Accumulation of a magnetic field in an imploding cylindrical shell is of special interest due to the feasibility of an experimental realization of such geometry and because in the first approximation such model describes an electromagnetic field generated in a gas compressed by a plasma shell.

Generally different configurations of the electromagnetic field can occur in a cylindrical geometry, namely, TM configuration with an electric field in the direction of a z axis, and a TE configuration with a magnetic field in the direction of z axis (see, e.g., Ref. 8, Chap. 10, Sec. 91). Solution of a boundary value problem for an inhomogeneous in the direction of z axis field when the boundary moves in the direction normal to the z axis is quite involved. Here we consider a more simple case when a field is homogeneous in the direction of z axis. Then a vector potential $\mathbf{A}(\mathbf{r},t)$ is a function of two spatial coordinates ρ and φ, and in transverse-electric (TE) and transverse-magnetic (TM) configurations it has A_φ and A_z components, respectively.

Vector potentials A_φ and A_z can be represented in the following form:

$$A_{z,\varphi}=\left[\frac{\rho(t)}{\rho(0)}\right]^\gamma \exp(\mathrm{im}\ \varphi)\Phi_{z,\varphi}\left[\frac{\rho(t)}{\rho(0)}\right], \tag{36}$$

where $\rho(t)$ and $\rho(0)$ are radii of a cylindrical shell at time t and $t=0$.

In a case of a toroidal vector potential $\mathbf{A}=(0,A_\varphi,0)$ condition $\nabla\cdot\mathbf{A}=0$ implies that $m=0$. Then equations for functions $\Phi_{z,\varphi}$ which can be derived from Eqs. (5) and (36) can be written as

$$\left(\frac{\partial^2}{\partial\chi^2}+\frac{1}{\chi}\frac{\partial}{\partial\chi}\right)\Phi_z-\frac{m^2}{\chi^2}\Phi_z=\beta^2\hat{S}\Phi_z,$$
$$\left(\frac{\partial^2}{\partial\chi^2}+\frac{1}{\chi}\frac{\partial}{\partial\chi}\right)\Phi_\varphi-\frac{1}{\chi^2}\Phi_\varphi=\beta^2\hat{S}\Phi_\varphi, \tag{37}$$

where $\chi=\rho(t)/\rho_0$ and an operator $\hat{S}$ is defined above [see Eq. (34)].

Boundary conditions in cylindrical geometry are as follows:

$$\gamma\Phi_z(1)=0,\quad (\gamma+1)\Phi_\varphi(1)=0. \tag{38}$$

Nonsingular at $\chi\to 0$ solutions of Eqs. (37) read

$$\Phi_z(\chi)=\chi^m F\left(\frac{s_z}{2},\frac{s_z-1}{2};m+1,\beta^2\chi^2\right),$$
$$\Phi_\varphi(\chi)=\chi F\left(\frac{s_\varphi}{2},\frac{s_\varphi-1}{2};2;\beta^2\chi^2\right), \tag{39}$$

where $s_z=m+1-\gamma$ and $s_\varphi=2-\gamma$. If function $\gamma(\beta)$ does not have singularities solutions (39) at $\beta\to 0$ coincide with solutions of Eq. (33) and conditions (38) are satisfied at $\gamma_z=0$ and $\gamma_\varphi=-1$.

Then Eqs. (36) and (38) determine variation of a vector potential during accumulation of a magnetic field while a behavior of the magnetic field is determined by the following formulas:

$$H_z=\frac{H_0}{\rho^2(t)}\left[F\left(\frac{3}{2},1;2;\beta^2\chi^2\right)+\frac{3}{2}\beta\chi F\left(\frac{5}{2},2;3;\beta^2\chi^2\right)\right], \tag{40}$$

$$H_\rho=\frac{H_0 m\ \exp(\mathrm{im}\ \varphi)}{\rho(t)}\chi^m A(\chi),$$
$$H_\varphi=-\frac{H_0 m\ \exp(\mathrm{im}\ \varphi)}{\rho(t)}\left[\chi^{m-1}A(\chi)+\frac{\beta}{2}\chi^{m+1}B(\chi)\right], \tag{41}$$

where $A(\chi)=F((m+1)/2,m/2;m+1;\beta^2\chi^2)$ and $B(\chi)=F((m+3)/2,(m+2)/2;m+2;\beta^2\chi^2)$.

Analysis of the behavior of the electromagnetic field in a case of electromagnetic waves is much more involved. In this case functions $\gamma(\beta)$ have singularities; and a representation of a solution through hypergeometric functions is not efficient. In order to determine the explicit dependence $\gamma(\beta)$ another representation of the solution, e.g., like Eq. (25) for a sphere is required. Since the formulas for the solution in this case are cumbersome we do not present them here.

V. CONCLUSIONS

The main goal of this study was to determine the temporal-spatial structure of the eigenmodes of a resonator with moving walls. Certainly the determined analytical formulas for these eigenmodes are not always valid, but they exhaust, probably, the class of relatively simple explicit solutions.

The obtained solutions allow one to determine the regions of validity of an adiabatic approximation when the frequency of the resonator follows the instantaneous values of geometrical parameters, and the amplitude is determined by a conservation of a magnetic flux.

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APPENDIX A

Here we demonstrate that solution (21) can be represented in the form given by Eq. (22). Formal substitution $1/c\,\partial/\partial t\to ik$ in Eq. (21) allows one to determine its solution in an explicit form (see, e.g., Ref. 10, Chap. 5, Sec. 33):

$$R_{k\ell}=r^\ell\left(\frac{d}{rdr}\right)^\ell\frac{\sin kr}{r}. \tag{A1}$$

After inverse substitution $k=i/c\,\partial/\partial t$ Eq. (A1) yields an operator equality:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\hat{R}}{\partial r}\right)-\frac{\ell(\ell+1)}{r^2}\hat{R}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\hat{R}, \tag{A2}$$

where

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$$\hat{R}=r^{\ell}\left(\frac{d}{r\,dr}\right)^{\ell}\frac{\sinh\left(\frac{r}{c}\frac{\partial}{\partial t}\right)}{r}. \tag{A3}$$

Taking into account that $\sinh(r/c\partial/\partial t)$ is a difference of two shift operators

$$\sinh\left(\frac{r}{c}\frac{\partial}{\partial t}\right)=\frac{1}{2}\left[\exp\left(\frac{r}{c}\frac{\partial}{\partial t}\right)-\exp\left(-\frac{r}{c}\frac{\partial}{\partial t}\right)\right]$$

and applying an operator equation (A2) to an arbitrary function $f(t)$ one arrives at Eq. (22).

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