Inductive instability in electrodynamic systems with rapidly varying parameters

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In this study we demonstrate the feasibility of generation of the electric charge and electric current in systems with rapidly varying parameters. It is shown that the electric current is excited spontaneously in the systems where electric resistance of their elements changes. Remarkably, effective damping resistance of the layered conductors may become negative, although ohmic resistance of its layers grows. Similar effects occur in electric circuits with different conductors connected in parallel. The effective resistance of such a system can decrease in spite of the increase of the ohmic resistance of the conductors.
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I. INTRODUCTION

It is shown that a wide class of the electrodynamic systems with rapidly varying parameters are unstable with respect to the spontaneous excitation of electric current. Examples of such systems are heterogeneous electric conductors comprising layers with different magnitudes of electric conductivity or electric circuits with parallel connection of electric impedances.

In this study we investigate the mechanism of the instability which is associated with the inductive effects. In $R-L$ circuits the instability can be caused either by a rapid variation of the electric conductivity or by rapid variations of inductance. In $R-C$ circuits instability occurs due to rapid variations of capacity or electric conductivity. It must be noted that, along with rapid variation of the parameters, there is another necessary condition for the excitation of the instability, namely, existence of the parallel sections of the electric circuit.

In our previous studies[1,2] it was shown that in a conductor composed of layers with different electric conductivities, motion of the interface separating these layers can cause both reduction or increase of the inductance of the conductor. If the rate of inductance decrease $\dot{L}$ is sufficiently high, the effective damping resistance of the conductor $R_d = R + \dot{L}$ (R is its ohmic resistance) can change sign and becomes negative. The latter situation corresponds to the transition of the system into an unstable state with respect to the spontaneous excitation of the electric current. In Ref. 3 we demonstrated that both a magnetostatic approximation and equations of electromagnetic diffusion yield the same equation for the velocity of the interface which provides the negative damping resistance. Thus a magnetostatic approximation can be employed at least for the qualitative description of these systems.

However, the analysis of the expression for the inductance of the conductor derived using the magnetostatic approximation[1] shows that the inductance of the conductor can also decrease in a case of a stationary position of the boundary separating domains with different conductivities due to the variation of electric conductivity of the layer. Depending upon the position of the boundary, the negative damping resistance can occur when conductivities of the layers decrease or increase.

The physical meaning of the latter result is quite transparent. Variation of the conductivities of the layers causes redistribution of the electric current over the cross section of the conductor. Since the total electric current I does not change while the magnetic field energy in the conductor changes, this is equivalent to the change of inductance of the system. If the inductance of the system decreases, the work is performed against the ponderomotive forces which, according to the Le-Chatelier principle, act to increase the inductance. Thus, energy from the external source, which provides the variation of the parameters of the system in the direction of inductance decrease, is transformed into the energy of a magnetic field. Such a process continues until the ponderomotive forces are not sufficient either to balance the external source or to perform the work over the ''third'' body to provide the increase of the inductance.

We do not consider here the nature of the processes which lead to the high rates of variation of the parameters of the system. Various phenomena can cause these variations, e.g., phase transitions, chemical reactions, ionization, and shock waves. In this connection it is of interest to note that, due to the temperature dependence of the work performed against the ponderomotive forces, it will always be the phase transition of the first kind provided that it is accompanied by the change of electric conductivity. In this study we determine only the rate of change of electric parameters in various systems which can also be used for analyzing other situations when the conditions for excitation of electric current are satisfied.

The article is organized as follows. In the Sec. II we present a general analysis of the dynamics of the electromagnetic field using Maxwell equations and demonstrate the existence of the current instability in a stationary medium. In Sec. III we determine conditions for excitation of the electric

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current using the magnetostatic approximation and approximation of the lumped-parameters system.

II. EXCITATION OF ELECTRIC CURRENT IN A STATIONARY MEDIUM

For general analysis of the instability consider an electric current with a density $\mathbf{j}(\mathbf{r},t)$ in a conducting medium which occupies a finite domain. At some distance from this domain the electric $\mathbf{E}(\mathbf{r},t)$ and magnetic $\mathbf{H}(\mathbf{r},t)$ fields induced by the current vanish

$$\int \mathbf{H}(\mathbf{r},t)d\mathbf{S}=\int \mathbf{E}(\mathbf{r},t)d\mathbf{S}=\int (\mathbf{E}\times\mathbf{H})d\mathbf{S}=0, \qquad (1)$$

where integration in Eq. (1) is performed over the infinitely removed surface.

The system of equations describing the dynamics of the electromagnetic fields and electric currents in the approximation of a magnetic diffusion (i.e., when the displacement current is much less than the ohmic current) reads

$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \quad \nabla\times\mathbf{H}=\frac{4\pi}{c}\mathbf{j}, \quad \frac{\mathbf{j}}{\sigma}=\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{H}, \qquad (2)$$

where $\mathbf{v}(\mathbf{r},t)$ is the velocity of the medium and $\sigma(\mathbf{r},t)$ is an electric conductivity (hereafter we use SGSM system of units).

The last equation in the system of Eqs. (2) describes Ohm's law for a moving medium (see, e.g., Ref. 4). Multiplying the first in Eqs. (2) by $\mathbf{H}$ and the second by $\mathbf{E}$ and using condition (1) after some algebra, we find that

$$-\frac{1}{8\pi}\frac{\partial}{\partial t}\int \mathbf{H}^2\, dV=\int \frac{\mathbf{j}^2}{\sigma}\, dV+\int \frac{\mathbf{v}}{c}(j\times\mathbf{H})dV. \qquad (3)$$

The latter equation shows that, if the work performed by a moving medium is directed against the ponderomotive forces, i.e., $\int\mathbf{v}/c(j\times\mathbf{H})dV<0$, and its absolute value is more than the Joule dissipation rate, then the energy of the magnetic field increases

$$\frac{\partial}{\partial t}\int \mathbf{H}^2\, dV>0. \qquad (4)$$

Condition (4) is accepted generally as a condition for the magnetic dynamo (see, e.g., Ref. 4, Chap. 6, Sec. 2). Note, that until now most of the studies investigated the magnetic dynamo associated with the fact that the magnetic lines are frozen into the flow of a conducting fluid. In the latter case the excitation of the magnetic dynamo requires rather complicated configurations of the velocity and magnetic fields and high Reynolds numbers $\mathrm{Re}\gg 1$. The magnetic dynamo which we analyzed in Ref. 3 is excited at relatively low Reynolds numbers $\mathrm{Re}\geq 1$, and occurs due to the independence of the fluid velocity upon the direction of an electric current. The most significant requirement is that the direction of the fluid velocity $\mathbf{v}$ coincides with that of the gradient of the electric conductivity. In contrast to most investigations of the magnetic dynamo (see, e.g., Ref. 4) in our study[3] we assumed that $\nabla\cdot\mathbf{v}\neq 0$ and that velocity at the surface $v_n\neq 0$. Such conditions occur in the conducting media with shock or rarefaction waves, fast phase transitions (melting or fusion), chemical reactions, etc.

The above analysis elucidates the difference between the dynamo which we studied in Ref. 3 and the dynamo which is studied commonly in magnetohydrodynamics.[4]

The main goal of the present study is to analyze a new type of the electric dynamo which is not a magnetic dynamo in the sense of our previous study[3] but it is accompanied by the increase of the total electric current in a conductor. The specific feature of this dynamo is that it can occur in a stationary medium with a macroscopic velocity $\mathbf{v}(\mathbf{r},t)=0$. At the same time this electromagnetic dynamo has an essentially nonstationary character and occurs in nonstationary heterogeneous conductors.

In order to demonstrate that the existence of this current dynamo does not contradict the general Eqs. (2) and (3), consider the total electric current in the system

$$I(t)=\int_S \mathbf{j}\, d\mathbf{S},$$

where the surface S depends on the geometry of the problem. Thus, e.g., in a cylindrical conductor S is its cross section. When an azimuthal electric current $\mathbf{j}=(0,0,j_\varphi)$ is excited in a conducting sphere with radius $\rho(t)$

$$I(t)=\int_0^{\rho(t)} j_\varphi\, d\theta\rho'\, d\rho'.$$

Define now an ohmic resistance $R(t)$ as

$$I^2R\equiv\int \frac{\mathbf{j}^2}{\sigma}\, dV \qquad (5)$$

and inductance $L(t)$ as

$$\frac{LI^2}{2}=\int \frac{H^2\, dV}{8\pi}. \qquad (6)$$

Then, when a macroscopic velocity $\mathbf{v}(\mathbf{r},t)=0$, Eqs. (3), (5), and (6) imply that

$$-\frac{L}{2}\frac{\partial}{\partial t}(I^2)=I^2\left(R+\frac{\partial L}{\partial t}\right). \qquad (7)$$

Therefore when

$$\left(R+\frac{\partial L}{\partial t}\right)<0 \qquad (8)$$

then $\partial/\partial t(I^2)>0$, although obviously $\partial/\partial t(LI^2/2)=-I^2R<0$. Thus the total magnetic energy decreases but the magnitude of the total electric current increases. This effect corresponds to the redistribution of the magnetic field whereby magnetic field concentrates in the current-carrying regions.

Thus formula (7) shows that increase of the total electric current without an external ac voltage source can also occur in a medium without hydrodynamic flows provided that the rate of the inductance change is sufficiently high. To the best of our knowledge, the latter fact was noted first in Ref. 1, where we studied the fast melting of a conductor which is accompanied by the rapid motion of the interface separating the regions with the different electric conductivities.

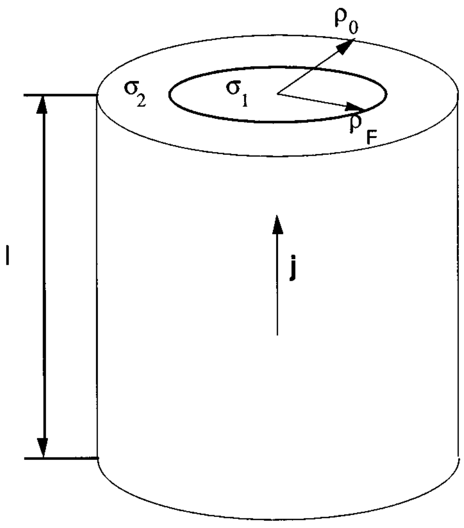

FIG. 1. Cylindrical conductor consisting of two concentric cylindrical layers with electric conductivities σ_1 and σ_2.

Analysis of the electric current instability using equations of magnetic diffusion is quite involved because it requires a solution of a nonstationary boundary value problem in a heterogeneous conducting medium.[3] Since only some self-similar solutions of such a problem can be determined analytically, in this study we analyze this problem using a magnetostatic approximation and an approximation of a lumped-parameters system (R, L, and C representation). Such investigation is of interest itself since it allows us to advance beyond the analysis of the self-similar situations.

III. NEGATIVE DAMPING RESISTANCE IN LAYERED CONDUCTORS

Determining a condition for the negative damping resistance of the conductor composed of layers with changing electric conductivity using Ohm's law for systems with varying inductance reads (see, e.g., Ref. 5, Chap. 7, Sec. 61)

$$L\dot{I}+I(R+\dot{L})=U_e,$$

where U_e is the external voltage and I is the total electric current. The condition for the instability in this system (8) is determined before. Consider a conductor composed of the cylindrical layers with the conductivity σ_1 for $0<\rho<\rho_F$ and with the conductivity σ_2 for $\rho_F<\rho<\rho_0$ (ρ_0 is the conductor's radius, see Fig. 1).

In the magnetostatic approximation we can use the following equations:

$$\text{rot } \mathbf{H}=\frac{4\pi}{c}\,\mathbf{j}, \quad \text{div } \mathbf{j}=0, \quad \mathbf{j}=\sigma\mathbf{E}. \tag{9}$$

Inductance of the conductor is determined by formula (6). Then, after some algebra, we arrive at the expression for the inductance of the conductor L (see Ref. 3):

$$L=L_0-\frac{l}{c^2}\,f(y,\kappa),$$

$$f(y,\kappa)=\xi^2y[y\,\ln(y)+\overline{a}(1-y)], \quad \overline{a}=\frac{\kappa}{\kappa-1}, \tag{10}$$

$$\kappa=\frac{\sigma_2}{\sigma_1}, \quad x=\frac{1}{\overline{a}-y},$$

where $L_0=2l/c^2\,\ln(l/\rho_0)$ is the inductance of a homogeneous conductor and $y=\rho_F^2/\rho_0^2\ll 1$. Electric resistance of a layered conductor $R=l/\overline{\sigma}\pi\rho_0^2$, $\overline{\sigma}=\sigma_1[y+\kappa(1-y)]$. Then Eqs. (8) and (10) yield a condition for a negative effective resistance of a single conductor

$$\left[\frac{2y}{\overline{a}-y}\,[y\,\ln(y)+\overline{a}(1-y)]-y(1-y)\right]\dot{\kappa}$$

$$>\frac{c^2}{\pi\rho_0^2\sigma_1}\,[y+\kappa(1-y)]. \tag{11}$$

Consider various limiting cases of the condition (11). Assume that the electric conductivity of the internal layer $\sigma_1\ll\sigma_2$, i.e., $\kappa\gg 1$, and also $y\ll 1$. Then

$$\dot{\kappa}>\frac{c^2}{\pi\rho_0^2\sigma_1}\,\frac{\kappa}{2y(y\,\ln(y)+1)}. \tag{12}$$

Thus in this range the instability occurs during the increase of the high specific electrical resistance of the internal layer or during the decrease of a smaller specific resistance of the external layer.

A different situation occurs in the range $y\ll\kappa\ll 1$. Here, instead of a condition (12) we obtain

$$\dot{\kappa}>\frac{c^2}{\pi\rho_0^2\sigma_1}\,\frac{\kappa^2}{2[\kappa-y\,\ln(y)]}.$$

In the range $\kappa\ll y\ll 1$ we obtain the following condition for the instability:

$$\dot{\kappa}>\frac{c^2}{\pi\rho_0^2\sigma_1}\,\frac{1}{\ln(e/y^2)}, \quad \ln(e)=1\,.$$

Thus in the conductor composed of parallel layers, rapid variation of its conductivity results in the instability of this conductor with respect to the excitation of the electric current. It is conceivable to suggest that a similar situation also occurs in the electric circuits with conductors connected in parallel. In the analysis of such systems it is possible to abandon the magnetostatic approximation and to account for the first order correction in the rate of change of the electrical current, i.e., to consider not only the ohmic but also an inductive voltage. To this end we consider the electric circuit shown in Fig. 2. This system is described by the following equations:

$$U=\hat{Z}_0I+\hat{Z}_1I_1,$$

$$\hat{Z}_1I_1=\hat{Z}_2I_2, \tag{13}$$

$$I=I_1+I_2,$$

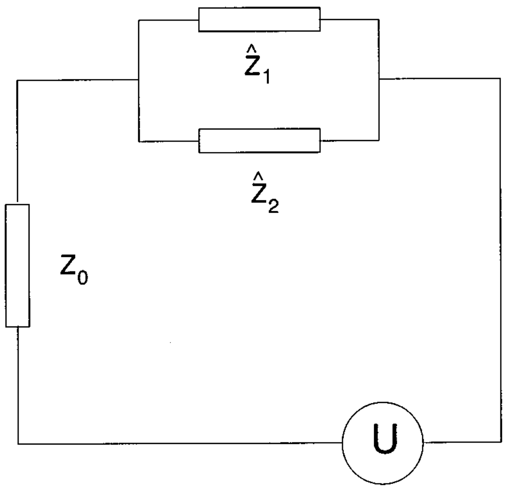

FIG. 2. Scheme of the electrical circuit.

where U is an external voltage and $\hat{Z}_i$ is an impedance operator $\hat{Z}_i = L_i \partial/\partial t + R_i(t)$. Eqs. (13) yield an equation for the dynamics of the total electric current I

$$U = \hat{Z}_0 I + \left(\hat{Z}_1 \frac{1}{\hat{Z}_1 + \hat{Z}_2} \hat{Z}_2 \right) I. \tag{14}$$

Analysis of the stability of the integro-differential Eq. (14) is rather involved. Therefore we adopt the simplifying assumptions that $R_1 \gg R_2$ and $R_1(t)$ is the only parameter varying with time. In this case it can be written

$$\hat{Z}_1 + \hat{Z}_2 = \tilde{L} \frac{\partial}{\partial t} + R_1(t), \quad \tilde{L} = L_1 + L_2. \tag{15}$$

Substituting Eq. (15) into Eq. (14) and introducing the new variable

$$\tau(t) = \int_0^t \frac{R_1(t')}{\tilde{L}} \, dt' \tag{16}$$

we arrive at the following equation:

$$\left(\frac{\partial}{\partial \tau} + 1 \right) \frac{U}{R_1(\tau)} = f \frac{\partial^2 I}{\partial \tau^2} + q(\tau) \frac{\partial I}{\partial \tau} + h(\tau) I, \tag{17}$$

where

$$f = \frac{L_1 L_2 + L_0 \tilde{L}}{\tilde{L}^2},$$

$$q(\tau) = \frac{R_1(L_2 + L_0) + L_1 R_2 + R_0 \tilde{L}}{\tilde{L} R_1}, \tag{18}$$

$$h(\tau) = \frac{R_0 + R_2}{R_1} + \left(R_0 + \frac{L_1}{\tilde{L}} R_2 \right) \frac{\partial}{\partial \tau} \left(\frac{1}{R_1} \right).$$

The condition for spontaneous excitation of electrical current can be determined by analyzing the system of basis solutions $I_1(t)$ and $I_2(t)$ of the differential Eq. (17). This condition reduces to the requirement for the existence of the indefinitely growing with time one of the basis solutions.

Estimate the condition for the spontaneous excitation of the electrical current using the WKB approximation (see, e.g., Ref. 6). In the zeroth WKB approximation the basis solutions $I_1(t)$ and $I_2(t)$ can be written as

$$I_{1,2}(\tau) = \exp\left\{ -\frac{1}{2} \int_{\tau_0}^{\tau} \frac{q(\tau')}{f} \, d\tau' \right\} \times \exp\left\{ \pm i \int_{\tau_0}^{\tau} \sqrt{\frac{s}{f} - \frac{q^2}{4f^2}} \, d\tau' \right\}, \tag{19}$$

where

$$s = \frac{R_0 + R_2}{R_1} + \frac{1}{2} \frac{R_0 \tilde{L} + L_1 R_2}{\tilde{L}} \frac{\partial}{\partial \tau} \left(\frac{1}{R_1} \right).$$

According to Eq. (19) the necessary condition for the excitation of the instability is $s < 0$. Returning to the real time the latter condition can be written as

$$\frac{\partial \ln(R_1)}{\partial t} > \frac{2(R_0 + R_2) R_1}{R_0 \tilde{L} + L_1 R_2}. \tag{20}$$

In the limit $R_2 \to 0$, the condition for the instability does not depend upon external parameter R_0.

A similar situation also occurs in $R-C$ circuits. All the obtained results are valid in this case provided that the equation for the impedance operator $\hat{Z}_i$ is replaced by $\hat{Z}_i = R_i \partial/\partial t + 1/C_i$. Certainly generation of the electrical charge instead of generation of electrical currents occurs in this case. The condition for the generation of the electrical charge in the system can be written as

$$\frac{\partial \ln(C_1)}{\partial t} < -\frac{2(C_2 + C_0)}{C_1(\tilde{R} C_2 + R_1 C_0)}, \quad \tilde{R} = R_1 + R_2. \tag{21}$$

Equation (21) describes the rate of reduction of electrical capacitance which provides generation of the electrical charge. Since ponderomotive forces in electrical capacitors are directed to increase the magnitude of the electric capacitance, their action causes ''annihilation'' of the noncompensated electrical charges.

Amplification of the transverse electromagnetic waves in resonators with a decreasing coefficient of dielectric permeability provides another example of the effects considered in this study. Assume that the characteristic dimensions of the resonator d satisfy the condition

$$\frac{c^2}{d^2} \gg \dot{\epsilon}^2 \sim \ddot{\epsilon}^2, \tag{22}$$

where ϵ is a coefficient of dielectric permeability.

In a plane resonator with time varying spatially homogeneous coefficient of the dielectric permeability $\epsilon(t)$ the Fourier transform of vector potential $\mathbf{A}_k(t)$ is determined by equation

$$\ddot{\mathbf{A}}_k + \frac{\dot{\epsilon}}{\epsilon} \dot{\mathbf{A}}_k - \frac{c^2 k^2}{\epsilon} \mathbf{A}_k = 0. \tag{23}$$

Fourier modes of the electric and magnetic fields are determined by

$$\mathbf{E}_k(t) = -\frac{1}{c}\frac{\partial \mathbf{A}_k}{\partial t}, \quad \mathbf{H}_k(t) = \mathbf{k}\times\mathbf{A}_k. \tag{24}$$

Provided that condition (22) is met and using WKB approximation, the solution of Eq. (23) can be written as follows:

$$\mathbf{A}_k = \{\mathbf{C}_1(\mathbf{k})\exp[i\phi(\mathbf{k},t)] + \mathbf{C}_2(\mathbf{k})\exp[-i\phi(\mathbf{k},t)]\}\epsilon^{-1/4}, \tag{25}$$

where

$$\phi = \int_{t_0}^{t} \frac{ck}{\epsilon^{1/2}(t')}\left(1+\frac{1}{4}\frac{\dot{\epsilon}^2 - 2\ddot{\epsilon}}{c^2k^2}\right)^{1/2} dt'. \tag{26}$$

Using formulas (24), coefficients $\mathbf{C}_1(\mathbf{k})$ and $\mathbf{C}_2(\mathbf{k})$ can be expressed through the fields $\mathbf{E}_k^0$ and $\mathbf{H}_k^0$ at the initial time t_0

$$\begin{aligned}\mathbf{E}_k(t) &= \frac{1}{\epsilon^{3/4}}\{\epsilon_0^{3/4}\mathbf{E}_k^0 \cos\phi + i\epsilon_0^{1/4}\mathbf{n}\times\mathbf{H}_k^0 \sin\phi\},\\ \mathbf{H}_k(t) &= \frac{1}{\epsilon^{1/4}}\{\epsilon_0^{1/4}\mathbf{H}_k^0 \cos\phi + i\epsilon_0^{3/4}\mathbf{E}_k^0\times\mathbf{n} \sin\phi\}.\end{aligned} \tag{27}$$

Electromagnetic energy of the kth mode of the resonator can be written as

$$W_k = \frac{\mathbf{D}(-\mathbf{k},t)\mathbf{E}(\mathbf{k},t)}{8\pi} + \frac{\mathbf{H}(\mathbf{k},t)H(-\mathbf{k},t)}{8\pi}, \tag{28}$$

where $\mathbf{D}(\mathbf{k},t) = \epsilon\mathbf{E}(\mathbf{k},t)$.

Equations (25)–(28) yield

$$W_k = \left(\frac{\epsilon_0}{\epsilon}\right)^{1/2} W_k^0, \tag{29}$$

where W_k^0 is energy in the kth mode at the initial time. Note that Eq. (29) can be obtained immediately since with the accuracy $O(\ddot{\epsilon}d^2/c^2)$ and $O(\dot{\epsilon}d^2/c^2)$, the ratio of the energy of the kth mode to its frequency W_k/ω_k, $\omega_k = ck/\epsilon^{1/2}$, is an adiabatic invariant (see, e.g., Ref. 5, Chap. 9, Sec. 81). Thus in the homogeneous resonator the adiabatically slow decrease of the dielectric permeability causes generation of the electromagnetic field.

IV. CONCLUSIONS

In this study we demonstrated the feasibility of the generation of the electric charge and electric current in systems with rapidly varying parameters.

Although the spontaneous excitation of the electric current in the systems where the ohmic resistance of their elements grows may seem unexpected, it can be explained simply. In the parallel circuits the total ohmic resistance is less than the smallest resistance of its elements. Therefore the increase of the larger resistances has little effect on the total resistance, although such change causes the significant redistribution of the electric current. Under the fixed magnitude of the total current, its redistribution is equivalent to the change of the inductance. If the rate of inductance variation is high enough the conditions for the excitation of electric current can be met.

The analyzed effects may have important technological applications and, to the best of our knowledge, have not been considered before. Certainly, the technological applications of the considered effects require to study the specific mechanisms which provide rapid variations of the parameters of the system.

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