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Diffusion of Low-Inertia Particles in a Field of Random Forces and the Kramers Problem

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Abstract—The diffusion of a low-inertia particle in the field of a random force (in the form of a random process) is studied using the simplest problem admitting an analytic solution as an example. The problem of determining the statistical characteristics of the particle's coordinates (the Kramers problem) is discussed in detail.

1. INTRODUCTION

Beginning with the classical work done by Stokes in 1851 [1] (see also the classical monograph [2]), the dynamics of inertial particles in hydrodynamic flows has attracted the attention of numerous researchers because of its importance for various ecological problems in the earth's atmosphere and ocean, as well as owing to a multiplicity of technical applications (see, for example, [3, 4]).

The diffusion of the field of the number density of inertial particles $n(\mathbf{r}, t)$ that move in random hydrodynamic flows described by an Eulerian velocity field $\mathbf{u}(\mathbf{r}, t)$ is controlled by the continuity equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\mathbf{V}(\mathbf{r}, t)\right)n(\mathbf{r}, t) = 0, \quad n(\mathbf{r}, 0) = n_0(\mathbf{r}), \tag{1}$$

and the total number of particles holds in the course of evolution; i.e.,

$$N_0 = \int n(\mathbf{r}, t)d\mathbf{r} = \int n_0(\mathbf{r})d\mathbf{r} = \text{const}.$$

The velocity field of low-inertia particles $\mathbf{V}(\mathbf{r}, t)$ will be described by the equation (see, for example, [5])

$$\left(\frac{\partial}{\partial t} + \mathbf{V}(\mathbf{r}, t)\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{V}(\mathbf{r}, t) = -\lambda[\mathbf{V}(\mathbf{r}, t) - \mathbf{u}(\mathbf{r}, t)], \tag{2}$$

which can be regarded as a phenomenological equation. The parameter $\tau = 1/\lambda$ is the well-known Stokes time, which depends on the particle size and the molecular viscosity. It follows from Eq. (2) that the velocity field $\mathbf{V}(\mathbf{r}, t)$ is a divergent field ($\operatorname{div}\mathbf{V}(\mathbf{r}, t) \neq 0$) even for a divergence-free hydrodynamic flow $\mathbf{u}(\mathbf{r}, t)$($\operatorname{div}\mathbf{u}(\mathbf{r}, t) = 0$). As a consequence, the clustering of the field of particle density in divergence-free hydrodynamic flows occurs [6–9].

If the parameter λ is large (low-inertia particles), when

$$\mathbf{V}(\mathbf{r}, t) \approx \mathbf{u}(\mathbf{r}, t), \tag{3}$$

the question arises as to what equality (3) means statistically and what the conditions of its applicability to statistical problems will be.

The first-order partial differential equations (1) and (2) (Eulerian description) are equivalent to the system of ordinary differential characteristic equations (Lagrangian description)

$$\begin{gathered}\frac{d}{dt}\mathbf{r}(t) = \mathbf{V}(\mathbf{r}(t), t), \quad \mathbf{r}(0) = \mathbf{r}_0,\\ \frac{d}{dt}\mathbf{V}(t) = -\lambda[\mathbf{V}(t) - \mathbf{u}(\mathbf{r}(t), t)],\\ \mathbf{V}(0) = \mathbf{V}_0(\mathbf{r}_0),\\ \frac{d}{dt}n(t) = -n(t)\frac{\partial \mathbf{V}(\mathbf{r}(t), t)}{\partial \mathbf{r}}, \quad n(0) = n_0(\mathbf{r}_0).\end{gathered} \tag{4}$$

We will assume that the variance of the random field of velocities $\sigma_{\mathbf{u}}^2 = \langle \mathbf{u}^2(\mathbf{r}, t)\rangle$ is sufficiently small and defines the main parameter of the problem. The first two equations control the diffusion of a particle under the effect of an external random force with linear friction. The low inertia of particles implies that the parameter $\lambda\tau_0 \gg 1$ even at a sufficiently small parameter τ_0, where $1/\lambda$ is the Stokes time and τ_0 is the correlation time of the initial incompressible ve locity field of the hydrodynamic flow $\mathbf{u}(\mathbf{r}, t)$. This means that the approximation of a velocity field delta-correlated in time is inapplicable to analysis of the problem, and the Fokker–Planck equation for the probability density function of particle velocity is invalid for low-inertia particles. The diffusion of low-inertia particles represents a fundamentally new class of statistical problems that have not been discussed in the scientific literature, although, in principle, we deal with a classical formulation of the problem. Statistical analysis of the nonlinear system of equations (4) for a Gaussian random field $\mathbf{u}(\mathbf{r}, t)$ with a finite correlation time can generally be performed in the context of the diffusion approximation (see, for example, [10]).

In the simplest case of a random force independent of the spatial coordinate, we have the system of equations

$$\frac{d}{dt}\mathbf{r}(t) = \mathbf{v}(t), \quad \frac{d}{dt}\mathbf{v}(t) = -\lambda[\mathbf{v}(t) - \mathbf{f}(t)], \qquad (5)$$
$$\mathbf{r}(0) = 0, \quad \mathbf{v}(0) = 0,$$

which will be considered in the following. The system of equations (5) admitting an analytic solution was discussed in a variety of textbooks (see, for example, [11]). In the limiting case of low-inertial particles ($\lambda \longrightarrow \infty$), however, almost all the textbooks contain erroneous statements associated with analysis of the statistical characteristics of a vector process $\mathbf{r}(t)$ in passing to the limit. We note that this problem corresponds to the so-called Kramers problem (see, for example, [12]).

2. STATISTICAL CHARACTERISTICS OF PARTICLE DYNAMICS

The solution to the stochastic equations (5) is written as

$$\mathbf{v}(t) = \lambda\int_0^t d\tau e^{-\lambda(t-\tau)}\mathbf{f}(\tau),$$
$$\mathbf{r}(t) = \int_0^t d\tau[1 - e^{-\lambda(t-\tau)}]\mathbf{f}(\tau). \qquad (6)$$

For a stationary random process $\mathbf{f}(t)$ with the correlation tensor $\langle f_i(t)f_j(t')\rangle = B_{ij}(t - t')$ and the correlation time τ_0 defined by

$$\tau_0 B_{ii}(0) = \int_0^\infty d\tau B_{ii}(\tau),$$

we obtain the following analytic expressions for the correlation functions of the particle velocity and coordinate:

$$\langle v_i(t)v_j(t)\rangle = \lambda\int_0^t d\tau B_{ij}(\tau)[e^{-\lambda\tau} - e^{-\lambda(2t-\tau)}],$$
$$\frac{1}{2}\frac{d}{dt}\langle r_i(t)r_j(t)\rangle = \langle r_i(t)v_j(t)\rangle \qquad (7)$$
$$= \int_0^t d\tau B_{ij}(\tau)[1 - e^{-\lambda t}][1 - e^{-\lambda(t-\tau)}].$$

In a stationary regime, when $\lambda t \gg 1$ and $t/\tau_0 \gg 1$, but the parameter $\lambda\tau_0$ is arbitrary, the particle velocity is a stationary process with the correlation tensor

$$\langle v_i(t)v_j(t)\rangle = \lambda\int_0^\infty d\tau B_{ij}(\tau)e^{-\lambda\tau}, \qquad (8)$$

and

$$\langle r_i(t)v_j(t)\rangle = \int_0^\infty d\tau B_{ij}(\tau),$$
$$\langle r_i(t)r_j(t)\rangle = 2t\int_0^\infty d\tau B_{ij}(\tau). \qquad (9)$$

If $\lambda\tau_0 \gg 1$, the correlation tensor switches to

$$\langle v_i(t)v_j(t)\rangle = B_{ij}(0) \qquad (10)$$

in accordance with Eq. (5), because $\mathbf{v}(t) \equiv \mathbf{f}(t)$ in this limiting case.

If $\lambda\tau_0 \ll 1$, then

$$\langle v_i(t)v_j(t)\rangle = \lambda\int_0^\infty d\tau B_{ij}(\tau),$$

and this result corresponds to the delta-correlated approximation of the random process $\mathbf{f}(t)$.

2.1. Probability Density Function

We introduce the indicator function for the solution of Eq. (5)

$$\phi(\mathbf{r}, \mathbf{v}; t) = \delta(\mathbf{r}(t) - \mathbf{r})\delta(\mathbf{v}(t) - \mathbf{v}),$$

which is described by the Liouville equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} - \lambda\frac{\partial}{\partial \mathbf{v}}\mathbf{v}\right)\phi(\mathbf{r}, \mathbf{v}; t) = -\lambda\mathbf{f}(t)\frac{\partial}{\partial \mathbf{v}}\phi(\mathbf{r}, \mathbf{v}; t), \qquad (11)$$
$$\phi(\mathbf{r}, \mathbf{v}; 0) = \delta(\mathbf{r})\delta(\mathbf{v}).$$

The indicator function $\phi(\mathbf{r}, \mathbf{v}; t)$ averaged over an ensemble of realizations of the random process $\mathbf{f}(t)$ describes the joint one-time probability density function of the particle location and velocity

$$P(\mathbf{r}, \mathbf{v}; t) = \langle\phi(\mathbf{r}, \mathbf{v}; t)\rangle$$
$$= \langle\delta(\mathbf{r}(t) - \mathbf{r})\delta(\mathbf{v}(t) - \mathbf{v})\rangle_{\mathbf{f}}.$$

If we average Eq. (11) over an ensemble of relaizations of the random process $\mathbf{f}(t)$, we obtain the nonclosed equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} - \lambda\frac{\partial}{\partial \mathbf{v}}\mathbf{v}\right)P(\mathbf{r}, \mathbf{v}; t) = -\lambda\frac{\partial}{\partial \mathbf{v}}\langle\mathbf{f}(t)\phi(\mathbf{r}, \mathbf{v}; t)\rangle, \qquad (12)$$
$$P(\mathbf{r}, \mathbf{v}; 0) = \delta(\mathbf{r})\delta(\mathbf{v}),$$

which contains the correlation $\langle \mathbf{f}(t)\phi(\mathbf{r}, \mathbf{v}; t)\rangle$. It is easy to see (see, for example, [10]) that Eq. (12) for the probability density function $P(\mathbf{r}, \mathbf{v}; t)$ is equivalent to the equality

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} - \lambda\frac{\partial}{\partial \mathbf{v}}\mathbf{v}\right)P(\mathbf{r}, \mathbf{v}; t)$$

$$= \left\langle \dot{\Theta}\left[t; \frac{\delta}{i\delta \mathbf{f}(\tau)}\right]\phi(\mathbf{r}, \mathbf{v}; t)\right\rangle, \tag{13}$$

$$P(\mathbf{r}, \mathbf{v}; 0) = \delta(\mathbf{r})\delta(\mathbf{v}),$$

where the functional $\dot{\Theta}[t; \mathbf{v}(\tau)]$ is related to the characteristic functional of the random process $\mathbf{f}(t)$

$$\Phi[t; \boldsymbol{\psi}(\tau)] = \left\langle \exp\left\{i\int_0^t d\tau \boldsymbol{\psi}(\tau)\mathbf{f}(\tau)\right\}\right\rangle = e^{\Theta[t;\, \boldsymbol{\psi}(\tau)]}$$

by the formula

$$\dot{\Theta}[t; \boldsymbol{\psi}(\tau)] = \frac{d}{dt}\ln\Phi[t; \boldsymbol{\psi}(\tau)] = \frac{d}{dt}\Theta[t; \boldsymbol{\psi}(\tau)].$$

The functional $\Theta[t; \boldsymbol{\psi}(\tau)]$ can be expanded into a functional power series:

$$\Theta[t; \boldsymbol{\psi}(\tau)] = \sum_{n=1}^{N}\frac{i^n}{n!}\int_0^t dt_1 \ldots \int_0^t dt_n K^{(n)}_{i_1, \ldots, i_n}$$
$$\times (t_1, \ldots, t_n)\psi_{i_1}(t_1)\ldots\psi_{i_n}(t_n),$$

where the functions

$$K^{(n)}_{i_1, \ldots, i_n}(t_1, \ldots, t_n) = \frac{1}{i^n}\frac{\delta^n}{\delta\psi_{i_1}(t_1)\ldots\delta\psi_{i_n}(t_n)}\Theta[v(\tau)]\big|_{v=0}$$

are the cumulant functions of the random process $\mathbf{f}(t)$ of order n.

Let us consider the variational derivative

$$\frac{\delta}{\delta f_j(t')}\phi(\mathbf{r}, \mathbf{v}; t)$$
$$= -\left[\frac{\partial}{\partial r_k}\frac{\delta r_k(t)}{\delta f_j(t')} + \frac{\partial}{\partial v_k}\frac{\delta v_k(t)}{\delta f_j(t')}\right]\phi(\mathbf{r}, \mathbf{v}; t). \tag{14}$$

For the dynamic problem (5), the variational derivatives of the functions $\mathbf{r}(t)$ and $\mathbf{v}(t)$ appearing in (14) are calculated from relations (6) and have the form

$$\frac{\delta v_k(t)}{\delta f_j(t')} = \lambda\delta_{kj}e^{-\lambda(t-t')}, \quad \frac{\delta r_k(t)}{\delta f_j(t')} = \delta_{kj}[1 - e^{-\lambda(t-t')}]. \tag{15}$$

Using equalities (15), one can write expression (14) in the form

$$\frac{\delta}{\delta \mathbf{f}(t')}\phi(\mathbf{r}, \mathbf{v}; t)$$
$$= -\left\{[1 - e^{-\lambda(t-t')}]\frac{\partial}{\partial \mathbf{r}} + \lambda e^{-\lambda(t-t')}\frac{\partial}{\partial \mathbf{v}}\right\}\phi(\mathbf{r}, \mathbf{v}; t).$$

Hence, Eq. (13) can be rewritten in a closed form:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} - \lambda\frac{\partial}{\partial \mathbf{v}}\mathbf{v}\right)P(\mathbf{r}, \mathbf{v}; t)$$

$$= \dot{\Theta}\left[t; i\left\{[1 - e^{-\lambda(t-t')}]\frac{\partial}{\partial \mathbf{r}} + \lambda e^{-\lambda(t-t')}\frac{\partial}{\partial \mathbf{v}}\right\}\right]P(\mathbf{r}, \mathbf{v}; t), \tag{16}$$

$$P(\mathbf{r}, \mathbf{v}; 0) = \delta(\mathbf{r})\delta(\mathbf{v}).$$

We note that the equations for the moment functions of order n that follow from Eq. (16) contain cumulant functions of an order not higher than n.

2.2. Case of a Gaussian Process $f(t)$

We assume that $\mathbf{f}(t)$ is a Gaussian stationary process with a zero mean and the correlation tensor

$$B_{ij}(t - t') = \langle f_i(t)f_j(t')\rangle.$$

In this case, the characteristic functional of the process $\mathbf{f}(t)$ is

$$\Phi[t; \boldsymbol{\psi}(\tau)]$$
$$= \exp\left\{-\frac{1}{2}\int_0^t\int_0^t dt_1 dt_2 B_{ij}(t_1 - t_2)\psi_i(t_1)\psi_j(t_2)\right\}.$$

For the functional $\dot{\Theta}[t; \boldsymbol{\psi}(\tau)]$, we obtain

$$\dot{\Theta}[t; \boldsymbol{\psi}(\tau)] = -\psi_i(t)\int_0^t dt' B_{ij}(t - t')\psi_j(t'),$$

and we arrive from Eq. (16) at the generalization of the Fokker–Planck equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} - \lambda\frac{\partial}{\partial \mathbf{v}}\mathbf{v}\right)P(\mathbf{r}, \mathbf{v}; t)$$

$$= \lambda^2\int_0^t d\tau B_{ij}(\tau)e^{-\lambda\tau}\frac{\partial^2}{\partial v_i \partial v_j}P(\mathbf{r}, \mathbf{v}; t) \tag{17}$$

$$+ \lambda\int_0^t d\tau B_{ij}(\tau)[1 - e^{-\lambda\tau}]\frac{\partial^2}{\partial v_i \partial r_j}P(\mathbf{r}, \mathbf{v}; t),$$

$$P(\mathbf{r}, \mathbf{v}; 0) = \delta(\mathbf{r})\delta(\mathbf{v}).$$

We note that the equation of the Fokker–Planck type (17) for the Gaussian process $\mathbf{f}(t)$ can also be derived immediately by averaging Eq. (11) and splitting the correlation on the right-hand side of Eq. (12) on the basis of the Furutsu–Novikov formula, which is written in the given case as

$$\langle f_i(t)R[t;\mathbf{f}(\tau)]\rangle = \int_0^t dt' B_{ij}(t-t')\left\langle \frac{\delta}{\delta f_j(t')}R[t;\mathbf{f}(\tau)]\right\rangle \quad (18)$$

and which is valid for the correlation of the Gaussian process $\mathbf{f}(t)$ with an arbitrary functional $R[t;\mathbf{f}(\tau)]$ $(0 \le \tau \le t)$ (see, for example, [10]). The variational derivatives of $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are described by (15).

Equation (17) is an exact equation and is valid for all times. It follows from it that the functions $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are Gaussian. In the usual way, we obtain the following system of equations for the moment functions of the processes $\mathbf{r}(t)$ and $\mathbf{v}(t)$:

$$\frac{d}{dt}\langle r_i(t)r_j(t)\rangle = 2\langle r_i(t)v_j(t)\rangle,$$

$$\left(\frac{d}{dt}+\lambda\right)\langle r_i(t)v_j(t)\rangle = \langle v_i(t)v_j(t)\rangle + \lambda\int_0^t d\tau B_{ij}(\tau)[1-e^{-\lambda\tau}], \quad (19)$$

$$\left(\frac{d}{dt}+2\lambda\right)\langle v_i(t)v_j(t)\rangle = 2\lambda^2\int_0^t d\tau B_{ij}(\tau)e^{-\lambda\tau}.$$

It follows from system (19) that if $\lambda t \gg 1$ and $t/\tau_0 \gg 1$, stationary values of all one-time correlations are described by

$$\langle v_i(t)v_j(t)\rangle = \lambda\int_0^\infty d\tau B_{ij}(\tau)e^{-\lambda\tau}, \quad \langle r_i(t)v_j(t)\rangle = D_{ij}, \quad (20)$$
$$\langle r_i(t)r_j(t)\rangle = 2tD_{ij},$$

where the spatial diffusion tensor has the form

$$D_{ij} = \int_0^\infty d\tau B_{ij}(\tau) \quad (21)$$

in accordance with (8) and (9).

Note. Time correlation tensor and correlation time of the process $\mathbf{v}(t)$.

We can also calculate the correlation time of the velocity $\mathbf{v}(t)$, i.e., the quantity $\langle v_i(t)v_j(t_1)\rangle$. Using (15) and (18), we obtain the equation for this quantity for times $t_1 < t$

$$\left(\frac{d}{dt}+\lambda\right)\langle v_i(t)v_j(t_1)\rangle = \lambda^2\int_0^{t_1} dt' B_{ij}(t-t')e^{-\lambda(t_1-t')}$$
$$= \lambda^2 e^{\lambda(t-t_1)}\int_{t-t_1}^{t} d\tau B_{ij}(\tau)e^{-\lambda\tau}, \quad (22)$$

which is subject to the initial condition

$$\langle v_i(t)v_j(t_1)\rangle|_{t=t_1} = \langle v_i(t_1)v_j(t_1)\rangle. \quad (23)$$

For a stationary regime when $\lambda t \gg 1$ and $\lambda t_1 \gg 1$, but $(t-t_1)$ is fixed, we obtain the equation with the initial condition:

$\tau = t - t_1$,

$$\left(\frac{d}{dt}+\lambda\right)\langle v_i(t+\tau)v_j(t)\rangle = \lambda^2 e^{\lambda\tau}\int_\tau^\infty d\tau_1 B_{ij}(\tau_1)e^{-\lambda\tau_1}, \quad (24)$$
$$\langle v_i(t+\tau)v_j(t)\rangle_{\tau=0} = \langle v_i(t)v_j(t)\rangle.$$

It is easy to write the solution of Eq. (24). However, we are interested in the correlation time τ_v of the random process $\mathbf{v}(t)$. To obtain this quantity, we integrate Eq. (24) with respect to the parameter τ over the interval $(0, \infty)$. As a result, we arrive at

$$\lambda\int_0^\infty d\tau\langle v_i(t+\tau)v_j(t)\rangle = \langle v_i(t)v_j(t)\rangle + \lambda\int_0^\infty d\tau_1 B_{ij}(\tau_1)[1-e^{-\lambda\tau_1}].$$

Hence, using (20), we obtain

$$\tau_v\langle \mathbf{v}^2(t)\rangle = D_{ii} = \tau_0 B_{ii}(0), \quad (25)$$

i.e.,

$$\tau_v = \frac{\tau_0 B_{ii}(0)}{\langle \mathbf{v}^2(t)\rangle} = \frac{\tau_0 B_{ii}(0)}{\lambda\int_0^\infty d\tau B_{ii}(\tau)e^{-\lambda\tau}} \quad (26)$$
$$= \begin{cases} \tau_0, & \text{if } \lambda\tau_0 \gg 1, \\ 1/\lambda, & \text{if } \lambda\tau_0 \ll 1. \end{cases}$$

Integrating Eq. (17) with respect to $\mathbf{r}$, we derive the closed-form equation for the probability density function of the particle velocity

$$\left(\frac{\partial}{\partial t} - \lambda\frac{\partial}{\partial \mathbf{v}}\mathbf{v}\right)P(\mathbf{v}; t)$$

$$= \lambda^2\int_0^t d\tau B_{ij}(\tau)e^{-\lambda\tau}\frac{\partial^2}{\partial v_i \partial v_j}P(\mathbf{r}, \mathbf{v}; t),$$

$$P(\mathbf{r}, \mathbf{v}; 0) = \delta(\mathbf{v}).$$

The solution of this equation corresponds to the Gaussian process $\mathbf{v}(t)$ with the correlation tensor (7). This is associated with the fact that the second equation of system (5) is closed. If $\lambda t \gg 1$, there is a stationary probability density function described by the equation

$$-\frac{\partial}{\partial \mathbf{v}}\mathbf{v}P(\mathbf{v}; t) = \lambda\int_0^\infty d\tau B_{ij}(\tau)e^{-\lambda\tau}\frac{\partial^2}{\partial v_i \partial v_j}P(\mathbf{v}; t),$$

and the rate at which this distribution reaches a stationary state depends on the parameter λ.

An equation for the probability density function of the particle coordinate $P(\mathbf{r}; t)$ cannot be obtained immediately from Eq. (17). Indeed, integrating Eq. (17) with respect to $\mathbf{v}$, we obtain the equality

$$\frac{\partial}{\partial t}P(\mathbf{r}, t) = -\frac{\partial}{\partial \mathbf{r}}\int \mathbf{v}P(\mathbf{r}, \mathbf{v}; t)d\mathbf{v}, \quad (27)$$

$$P(\mathbf{r}; 0) = \delta(\mathbf{r}).$$

For the function $\int v_k P(\mathbf{r}, \mathbf{v}; t)d\mathbf{v}$, we have the equality

$$\left(\frac{\partial}{\partial t} + \lambda\right)\int v_k P(\mathbf{r}, \mathbf{v}; t)d\mathbf{v} = -\frac{\partial}{\partial \mathbf{r}}\int v_k \mathbf{v}P(\mathbf{r}, \mathbf{v}; t)d\mathbf{v}$$
$$-\lambda\int_0^t d\tau B_{kj}(\tau)[1 - e^{-\lambda\tau}]\frac{\partial}{\partial r_j}P(\mathbf{r}, t), \quad (28)$$

etc. Thus, we obtain an infinite system of equations.

The random function $\mathbf{r}(t)$ is described by the first equation of system (5), and if the full statistics of the function $\mathbf{v}(t)$ (i.e., many-time statistics) were known, we could calculate all the statistical characteristics of the function $\mathbf{r}(t)$. However, Eq. (17) describes one-time statistical quantities alone, and only an infinite system of quantities similar to (27), (28), etc. is equivalent to many-time statistics for $\mathbf{v}(t)$. Indeed, the function

$$\mathbf{r}(t) = \int_0^t dt_1 \mathbf{v}(t_1),$$

and, as was seen in (25), the spatial coefficient of diffusion in a stationary regime equals

$$\frac{1}{2}\frac{d}{dt}\langle \mathbf{r}^2(t)\rangle = \int_0^\infty d\tau\langle \mathbf{v}(t+\tau)\mathbf{v}(t)\rangle \quad (29)$$
$$= \tau_v\langle \mathbf{v}^2(t)\rangle = D_{ii} = \tau_0 B_{ii}(0),$$

and depends on the correlation time τ_v of the random function $\mathbf{v}(t)$ and on its variance.

For this simplest problem, however, we immediately know the variances of all the quantities and the correlation functions of $\mathbf{v}(t)$ and $\mathbf{r}(t)$ (see expressions in (7)) and, consequently, we can write an equation for the probability density function of the particle coordinate $P(\mathbf{r}; t)$, which is the diffusion equation

$$\frac{\partial}{\partial t}P(\mathbf{r}, t) = D_{ij}(t)\frac{\partial^2}{\partial r_i \partial r_j}P(\mathbf{r}, t),$$

$$P(\mathbf{r}, 0) = \delta(\mathbf{r})$$

with the diffusion tensor (7)

$$D_{ij}(t) = \frac{1}{2}\frac{d}{dt}\langle r_i(t)r_j(t)\rangle$$

$$= \frac{1}{2}\{\langle r_i(t)v_j(t)\rangle + \langle r_j(t)v_i(t)\rangle\}$$

$$= \int_0^t d\tau B_{ij}(\tau)[1 - e^{-\lambda\tau}][1 - e^{-\lambda(t-\tau)}].$$

If $\lambda t \gg 1$, we obtain the equation

$$\frac{\partial}{\partial t}P(\mathbf{r}; t) = D_{ij}\frac{\partial^2}{\partial r_i \partial r_j}P(\mathbf{r}, t), \quad P(\mathbf{r}, 0) = \delta(\mathbf{r}) \quad (30)$$

with the diffusion tensor (29).

We note that switching from Eq. (17) to the equation for the probability density function of the particle coordinate (30) with the diffusion coefficient (29) corresponds to the Kramers problem mentioned above.

2.2.1. Delta-Correlated Approximation ($\lambda\tau_0 \ll 1$)

If $\lambda\tau_0 \ll 1$, where τ_0 is the correlation time of the process $\mathbf{f}(t)$, Eq. (17) is simplified to

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} - \lambda\frac{\partial}{\partial \mathbf{v}}\mathbf{v}\right)P(\mathbf{r}, \mathbf{v}; t)$$

$$= \lambda^2\int_0^t d\tau B_{ij}(\tau)\frac{\partial^2}{\partial v_i \partial v_j}P(\mathbf{r}, \mathbf{v}; t),$$

$$P(\mathbf{r}, \mathbf{v}; 0) = \delta(\mathbf{r})\delta(\mathbf{v}),$$

and corresponds to the so-called approximation of a delta-correlated process for the random function $\mathbf{f}(t)$. If the time $t \gg \tau_0$, we can replace the upper limit in the integral by infinity and can switch to the conventional Fokker–Planck diffusion equation

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}-\lambda\frac{\partial}{\partial \mathbf{v}}\mathbf{v}\right)P(\mathbf{r},\mathbf{v};t) = \lambda^2 D_{ij}\frac{\partial^2}{\partial v_i \partial v_j}P(\mathbf{r},\mathbf{v};t), \tag{31}$$
$$P(\mathbf{r},\mathbf{v};0)=\delta(\mathbf{r})\delta(\mathbf{v}),$$

with the diffusion tensor (29). In this approximation, the combined random process $\{\mathbf{r}(t), \mathbf{v}(t)\}$ is a Markov process.

If $\lambda t \gg 1$, there is a stationary equation for the probability density function of the particle velocity:

$$-\lambda\frac{\partial}{\partial \mathbf{v}}\mathbf{v}P(\mathbf{v}) = \lambda^2 D_{ij}\frac{\partial^2}{\partial v_i \partial v_j}P(\mathbf{v})$$

and a nonstationary equation for the probability density function of the particle coordinate:

$$\frac{\partial}{\partial t}P(\mathbf{r};t) = D_{ij}\frac{\partial^2}{\partial r_i \partial r_j}P(\mathbf{r};t), \quad P(\mathbf{r};0)=\delta(\mathbf{r}).$$

2.2.2. Another Asymptotic Case ($\lambda\tau_0 \gg 1$)

Let us consider the limiting case $\lambda\tau_0 \gg 1$, when Eq. (17) can be written as

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}-\lambda\frac{\partial}{\partial \mathbf{v}}\mathbf{v}\right)P(\mathbf{r},\mathbf{v};t)$$
$$= \lambda B_{ij}(0)[1-e^{-\lambda\tau}]\frac{\partial^2}{\partial v_i \partial v_j}P(\mathbf{r},\mathbf{v};t)$$
$$- B_{ij}(0)[1-e^{-\lambda\tau}]\frac{\partial^2}{\partial v_i \partial r_j}P(\mathbf{r},\mathbf{v};t)$$
$$+\lambda\int_0^t d\tau B_{ij}(\tau)\frac{\partial^2}{\partial v_i \partial r_j}P(\mathbf{r},\mathbf{v};t),$$
$$P(\mathbf{r},\mathbf{v};0)=\delta(\mathbf{r})\delta(\mathbf{v}).$$

Integrating this equation with respect to $\mathbf{r}$, we obtain the equation for the probability density function of the particle velocity

$$\left(\frac{\partial}{\partial t}-\lambda\frac{\partial}{\partial \mathbf{v}}\mathbf{v}\right)P(\mathbf{v};t) = \lambda B_{ij}(0)[1-e^{-\lambda t}]\frac{\partial^2}{\partial v_i \partial v_j}P(\mathbf{v};t),$$
$$P(\mathbf{v};0)=\delta(\mathbf{v}).$$

If $\lambda t \gg 1$, we find a stationary Gaussian probability density function with the variance

$$\langle v_i(t)v_j(t)\rangle = B_{ij}(0).$$

As for the probability density function of the particle coordinate, if $\lambda t \gg 1$, we obtain the equation

$$\frac{\partial}{\partial t}P(\mathbf{r};t) = D_{ij}\frac{\partial^2}{\partial r_i \partial r_j}P(\mathbf{r},t), \quad P(\mathbf{r},0)=\delta(\mathbf{r}) \tag{32}$$

with the same diffusion coefficient as previously. This is associated with equality (25), which is independent of the parameter λ. We note that this equation corresponds to the passage to the limit $\lambda \longrightarrow \infty$ in Eq. (5):

$$\frac{d}{dt}\mathbf{r}(t)=\mathbf{v}(t), \quad \mathbf{v}(t)=\mathbf{f}(t), \quad \mathbf{r}(0)=0.$$

Note. Limit $\lambda \longrightarrow \infty$ and the Kramers problem

In the limiting case $\lambda \longrightarrow \infty$ ($\lambda\tau_0 \gg 1$), we have

$$\mathbf{v}(t) \approx \mathbf{f}(t), \tag{33}$$

and all many-time statistics of the random functions $\mathbf{v}(t)$ and $\mathbf{r}(t)$ will be described by the statistical characteristics of the process $\mathbf{f}(t)$. Specifically, the one-time probability density function of the particle velocity $\mathbf{v}(t)$ is a Gaussian function with the variance $\langle v_i(t)v_j(t)\rangle = B_{ij}(0)$, and the spatial tensor of diffusion is defined by

$$D_{ij} = \frac{1}{2}\frac{d}{dt}\langle r_i(t)r_j(t)\rangle = \int_0^\infty d\tau B_{ij}(\tau).$$

Let the delta-correlated approximation of the process $\mathbf{f}(t)$ be valid ($\lambda\tau_0 \ll 1$). As was seen previously, the approximate equality (33) is no longer valid for finding the statistical characteristics of the process $\mathbf{v}(t)$. However, for the one-time statistical characteristics of the process $\mathbf{r}(t)$, we have Eq. (32) as previously with the same spatial diffusion tensor. This is due to equality (25), which is valid for any value of the parameter λ and an arbitrary probability density function of the random process $\mathbf{f}(t)$. Therefore, in the Kramers problem, the following statement will be erroneous: If the delta-correlated approximation is valid for the random process $\mathbf{f}(t)$, the statistics of the process $\mathbf{r}(t)$ for $\lambda t \gg 1$ will be described by Eq. (32) owing to equality (33). In the physical literature, such statements are usually met in discussing problems associated with the establishment of stationary Hibbs, Maxwell, and Boltzmann distributions.

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