A stationary principle for non conservative systems

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A stationary principle is described to yield governing integral formulations for dissipative systems. Variation is applied on selective terms of energy or momentum functionals resulting with force or mass balance equations respectively. Applying the principle for a motion of a viscous fluid yields the Navier–Stokes equations as an approximation of the functional (i.e. equating to zero part of the integrand). When a Darcy's flow regime in a porous media is considered, implementing a space averaging method on the resultant integral derived by the principle, Forchheimer's law for energy accumulation and solute transport equation for momentum assembling are yielded in differential form approximation of a more extended functional formulation.

Key Words: Hamilton's extended principle, dissipative systems, energy, momentum, Darcy flow regime, spatial averaging, minimum criterion.

INTRODUCTION

Equations governing physical phenomena are habitually written in a differential form. Such equations are derived by a summation of driving generalized forces upon an infinitesimal region assumed continuous. Thus information that is allocated within the area of interest surrounding the infinitesimal region is abandoned.[1,2]

An integral formulation can be extended to a larger spatial scale and thus would incorporate information in the space-time domain (e.g. prescribed functions relating properties to their spatial distribution). On the other hand, the numerical solution of highly dynamical systems is much more stable when an integral formulation (compared to a differential form) is implemented.[3]

In refs. 4 and 5 an energy postulate is suggested to include also systems with dissipation. However, when applied for irreversible processes it seems physically obscure and cumbersomely implemented.

Compatible with Hamilton's extended principle,[3] when applied to a continuous media, a postulate is herein derived introducing a variation on the time integral applied over the accumulation of power terms within a control volume. Carrying out a total time derivation on rigorously presented energy terms and adding a force dissipation junction, yields an Euler–Lagrange equation of motion as the integrand. However, when proceeding with the variation and upon inserting a trial function into the functional, a formulation that of searching an extremum is obtained. An extension is then demonstrated when the principle is applied on terms representing momentum and its dissipative function. The resultant functional describes an integral form of a mass transport equation.

THEORY

Hamilton's extended principle for non-conservation dynamical systems states that[3]

$$\delta H = \delta \int_{t_1}^{t_2} L(\mathbf{x}, \mathrm{v}, t)\, \mathrm{d}t + \int_{t_1}^{t_2} \bar{\bar{\mathbf{f}}} \cdot \delta \mathbf{x}\, \mathrm{d}t = 0 \qquad (1)$$

Here, δ, variational operator; $[t_1, t_2]$, time interval; $L = T - \Phi$, the Lagrangian where T and Φ are the kinetic and potential energies respectively; $\mathbf{f}$, vector of generalized dissipation forces; $\mathbf{x}$, vector of generalized co-ordinates; $\mathrm{v} = D\mathbf{x}/Dt$, vector of absolute generalized velocities; D/Dt, total time derivative. Notice that in (1) not all terms underwent a variation (such terms are denoted by $(\bar{\bar{\ }})$). Thus it is regarded as a stationary rather than a minimum principle. Upon carrying on the variation in (1) the Euler–Lagrange equation of motion is yielded:

$$\frac{D}{Dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{x}} = \mathbf{f} \qquad (2)$$

It will now be shown that equation (2) can be achieved by a stationary postulate defined over a space domain. Consider the following principle:

$$\delta J = \delta \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{D}{Dt} \int_{V(t)} \rho L^* \delta\, \mathrm{d}V \right] + \int_{t_1}^{t_2} \mathrm{d}t \left[\int_{V(t)} \bar{\bar{\mathbf{f}}}^* \cdot \delta \frac{D\mathbf{x}}{Dt} \rho\, \mathrm{d}V \right] = 0 \qquad (3)$$

Here $\mathrm{d}V$ is a volume element from $V(t)$. L^* and f^* are specific values of L, f respectively; ρ, specific mass. Keeping $D\mathbf{v}/Dt$ constant during $[t_1, t_2]$, integration by parts of (3) yields:

$$\delta J = -\delta \int_{t_1}^{t_2} \int_V \left[\frac{D}{Dt}\left(\frac{\partial L^*}{\partial \mathbf{v}} - \frac{\partial L^*}{\partial \mathbf{x}} \right) \bar{\bar{\mathbf{f}}} \right] \cdot \mathbf{v} \rho\, \mathrm{d}V\, \mathrm{d}t + \delta \int_{V_1}^{V_2} \left[\rho L^* + \rho \frac{\partial L^*}{\partial \mathbf{v}} \cdot \mathbf{v} \right] \Bigg|_{\tau(V)} \mathrm{d}V = 0 \qquad (4)$$

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where τ the inverse limit functions of $V(t)$, V_1 and V_2 limit values of V corresponding to $[t_1, t_2]$. Thus upon fulfilling the natural boundary condition in (4) the remainder is similar to (2) when part of the integrand is equated to zero. Hence the validity of postulate (3) is established.

IMPLEMENTATION

Motion of viscous fluid

The realization of (3) yields:

$$\delta J = \delta \int_{t_1}^{t_2} dt \left\{ \frac{D}{Dt} \int_V \left[\frac{1}{2} v^2 - (U + \Phi) \right] \right\} \rho \, dV$$

$$+ \int_{t_1}^{t_2} dt \left[\int_V \bar{\bar{\mathbf{f}}}^* \cdot \delta \mathbf{v} \rho \, dV \right] = 0 \qquad (5a)$$

Here U and Φ are internal and potential energies per unit mass respectively. Equation (5a) is thus taking the form:

$$\delta J = \delta \int_{t_1}^{t_2} \int_V \left[\rho \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} - \rho \frac{DU}{Dt} - \rho \frac{D\Phi}{Dt} + \rho \bar{\bar{f}} \mathbf{v} \right] dV \, dt = 0 \qquad (5b)$$

In view of the equation describing mass conservation:

$$\frac{D}{Dt} \rho = - \rho \nabla \cdot \mathbf{v} \qquad (6)$$

the change of the internal energy per unit time is:

$$\rho \frac{DU}{Dt} = \rho \frac{DQ}{Dt} - \rho \frac{DW}{Dt} = \rho \theta \frac{DS}{Dt} - P \nabla \cdot \mathbf{v} \qquad (7)$$

where Q, quantity of heat; W, mechanical work; θ, temperature; S, entropy; and P the pressure. Notice now that for an inviscid fluid no heat loss occurs, equation (7) takes the form:

$$\frac{DU}{Dt} = - P \nabla \cdot \mathbf{v} \qquad (8a)$$

In such a fluid no dissipation takes places, therefore

$$f = 0 \qquad (8b)$$

However, for a viscous fluid the kinematic heat loss yield the rise of internal stresses σ. Equation (7) is thus rewritten to:

$$\rho \frac{DU}{Dt} = - (\sigma + P) \nabla \cdot \mathbf{v} \qquad (9a)$$

The corresponding dissipation force, derived from the internal stresses is hence:

$$f = 2 \nabla \sigma \qquad (9b)$$

Notice the appearance of the coefficient in (9b), it is chosen so that the resultant between the dissipation force and the force derived from heat loss equals to one unit of $\nabla \sigma$. Thus in view of (8b), (9b) and above, the dissipation force takes the general form of:

$$f = r \cdot \nabla \sigma; \qquad r = 0, 1, 2 \qquad (10)$$

Substituting (9) into (5b), applying the time total derivative to Φ ($\Phi = \Phi(\mathbf{x})$) and with the aid of integration by parts, one yields the form:

$$\delta J = - \delta \int_{t_1}^{t_2} \int_V \left[\rho \frac{D\mathbf{v}}{Dt} + \nabla P + \rho \nabla \Phi - \overline{\overline{\nabla \sigma}} \right] \cdot \mathbf{v} \, dV \, dt$$

$$+ \delta \int_V^{V_2} [\rho v^2] \Big|_{\tau(V)} dV + \delta \int_{t_1}^{t_2} \int_{A(t)} (\sigma + P) \mathbf{v} \cdot \mathbf{n} \, dA \, dt = 0 \qquad (11)$$

where A is the surface area closing V, and n is a unit vector perpendicular to the surface and directed outwards.

With the fulfilment of the natural boundary conditions one attains the differential form of N.S. equations:

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla P + \rho \nabla \Phi - \nabla \sigma = 0 \qquad (12)$$

However, a more extended integral form is represented by the functional:

$$J_1 = \int_{t_1}^{t_2} \int_V \left[\rho \frac{D\mathbf{v}}{Dt} + \nabla P + \rho \nabla \Phi - \overline{\overline{\nabla \sigma}} \right] \cdot \mathbf{v} \, dV \, dt \quad (13a)$$

Assuming a trial function based on k variables, β_k allocated at prescribed locations in the time space domain, when placing the parameters in (13a), a function with several independent variables is yielded. Proceeding with the variation, a formulation that of searching an extremum is obtained:

$$\frac{\partial J_1}{\partial \beta_k} = 0 \qquad (13b)$$

The assembling of sets of equations like (13b) about all allocations results with a global system of algebraic equation. The solution of the set yields the distribution of governing variables in the domain during the time interval $[t_1; t_2]$.

Steady motion without vorticity of a fluid in a Darcy flow regime

It will be given here in the derivation of integral governing equations for the motion of a viscous fluid in a Darcy flow regime. It will also be shown that when a flow through a porous media is considered, a mean functional is attained by means of spatial averaging.

Consider the constitutive law relating stresses to the rate of strains in fluids:

$$\sigma_{ij} = 2\mu \dot{\epsilon}_{ij} - (\tfrac{2}{3}\mu \nabla \cdot \mathbf{v} + P) \delta_{ij} \qquad (\sigma_{ij} = \sigma_{ji}) \quad (14a)$$

Here, δ_{ij}, the kronical delta and

$$\dot{\epsilon}_{ij} = \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial \xi_j}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} \right) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \qquad (14b)$$

where ξ denotes an infinitesimal displacement. Also let

$$\Phi = gZ \qquad (15)$$

g, gravitational acceleration; Z, altitude.

$$\gamma = \rho g \qquad (16a)$$

$$h = \frac{P}{\gamma} + Z \qquad (16b)$$

γ, specific weight; h, hydraulic head.

Substituting equations (10), (14) and (15) into (5b) and by taking into account that $\partial / \partial t = 0$ and ρ = constant, equation (13a) yields:

$$J_2=\int_{t_1}^{t_2}\int_V[\rho\mathbf{v}\cdot\nabla\mathbf{v}+\gamma\nabla Z+(\overline{\overline{\nabla P-\mu\nabla^2\mathbf{v}}})]\cdot\mathbf{v}\,\mathrm{d}V\,\mathrm{d}t \quad (17a)$$

Notice that for very slow motion the first term in (17a) can be neglected to yield:

$$J_3=\int_{t_1}^{t_2}\int_V[\gamma\nabla Z+(\nabla P-\mu\nabla^2\mathbf{v})]\cdot\mathbf{v}\,\mathrm{d}V\,\mathrm{d}t \quad (17b)$$

When a flow through a porous media is considered the media can be divided into sections with average properties evaluated with respect to local spatial averaging.

Let $\bar{\mathbf{q}}$ be the total average vector of specific discharge that corresponds to total space average velocity $\bar{\mathbf{v}}$ by:

$$\bar{\mathbf{q}}=\eta\bar{\mathbf{v}}=\sum_m\eta_m\bar{\mathbf{v}}_m \quad (18a)$$

Here η the total mean porosity defined as the ratio between the total volume of voids, V_v, and the total summation of the voids and solid volumes, V_s.

$$\eta=\frac{V_v}{V}=\frac{V_v}{V_s+V_v} \quad (18b)$$

i.e.

$$\eta_m=\left(\frac{V_v}{V_s+V_v}\right)_m \quad (18c)$$

By a similar average procedure as described in ref. 6, equation (17a) is rewritten to the form:

$$\bar{J}_2=\int_{t_1}^{t_2}\sum_m\int_{V_m}[a\mathbf{A}+\gamma\nabla Z+(\overline{\overline{\nabla P+b\mathbf{B}}})]\cdot\left(\frac{q}{\eta}\right)\mathrm{d}V\,\mathrm{d}t \quad (19a)$$

where

$$a=\left(\frac{1-\eta}{\eta^3}\cdot\frac{q^2}{gV_s}\right)_m=(\bar{a}\bar{q}^2)_m \quad (19b)$$

$$b=\frac{(1-\eta)^2}{\eta^3}\cdot\frac{\mu/\rho}{gV_s^2}\cdot|\bar{q}|_m=(\bar{b}|\bar{q}|)_m \quad (19c)$$

$\mathbf{A}$, $\mathbf{B}$ are vectors taking into account cosine directions and shape factors of the pores.

Thus, by the procedure described in equation (13a) and in the lines above it, a set of algebraic equations extending throughout the whole media is established. It is still left to notice that when the variation is applied on (19), by virtue of (16a) and by equating the integrand between the parentheses to zero, a subset less extensive form of differential equation is achieved, namely Forchheimer law:

$$\bar{a}q^2\mathbf{A}+\bar{b}|\bar{q}|\mathbf{B}=-\nabla h \quad (20)$$

Solute transport in a Darcy flow regime

Herein it will be demonstrated that when applying the established stationary principle on the momentum and its related dissipative function, a mass transport equation results. When a flow through a porous media is considered by means of space averaging, a mean functional describing the advection-dispersion in an integral form is attained.

Let $\mathbf{P}=C\mathbf{v}$ denote the momentum of a solvent with concentration C moving with an instantaneous fluid velocity of $\mathbf{v}$. The withdrawal of $\mathbf{P}$ from a volume V enclosed by a surface A, would (by the divergence theorem) be:

$$\int_A(c\mathbf{v})\cdot\mathbf{n}\,\mathrm{d}A=\int_A\nabla\cdot(c\mathbf{v})\,\mathrm{d}V=\int_V[\mathbf{v}\cdot\nabla c+c\nabla\cdot\mathbf{v}]\,\mathrm{d}V \quad (21)$$

Implementing the stationary principle of (3) on the momentum confined within V is thus:

$$\delta I=\delta\int_{t_1}^{t_2}\mathrm{d}t\left[\frac{D}{Dt}\int_V\nabla\cdot\mathbf{P}\,\mathrm{d}V\right]+\int_{t_1}^{t_2}\mathrm{d}t\left[\int_V\bar{\bar{F}}\delta\frac{DC}{Dt}\,\mathrm{d}V\right]=0$$

$$(22)$$

Here F is the dissipation function chosen to be a function of velocity

$$F(v)=-|\mathbf{v}| \quad (23)$$

Thus in view of (21) and (23) equation (22) takes the form:

$$\delta I=\delta\int_{t_1}^{t_2}\left[\frac{D}{Dt}\int_V(\alpha\mathbf{v}\cdot\nabla c+\alpha c\nabla\cdot\mathbf{v})\,\mathrm{d}V\right]\mathrm{d}t$$

$$-\int_{t_1}^{t_2}\int_V|\bar{\mathbf{v}}|\delta\frac{DC}{\mathrm{d}t}\,\mathrm{d}V\,\mathrm{d}t=0 \quad (24)$$

Here α is a length factor.

For a constant density by virtue of (6) and by denoting

$$\mathbf{d}=\alpha\mathbf{v}$$

equation (24) with a constant acceleration is rewritten to be:

$$\delta I=-\delta\int_{t_1}^{t_2}\int_V|\bar{\mathbf{v}}|\left[\frac{DC}{Dt}-\mathbf{v}\cdot\nabla(\mathbf{d}\cdot\nabla c)\right]\mathrm{d}V\,\mathrm{d}t$$

$$+\delta\int_V\nabla\cdot(\mathbf{d}c)\,\mathrm{d}V=0 \quad (26)$$

In view of the divergence theorem, the fulfilment of the natural boundary condition denoted by I_0 states that:

$$\delta I_0=\delta\int_V\nabla\cdot(c\mathbf{d})\,\mathrm{d}V=\delta\int_A(c\mathbf{d})\cdot\mathbf{n}\,\mathrm{d}A=0 \quad (27)$$

Denoting I_1 as the functional

$$I_1=\int_{t_1}^{t_2}\int_V\left[|\bar{\mathbf{v}}|\frac{DC}{Dt}-\mathbf{v}\cdot\nabla(\mathbf{d}\cdot\nabla c)\right]\mathrm{d}V\,\mathrm{d}t \quad (28a)$$

By virtue of (26) and (27) one can thus write:

$$\delta I_1=0 \quad (28b)$$

Consider now a Darcy flow through a porous media, uncertainty embedded in such a zone can be regarded as a composite of spatial mean values and such that represent deviation from the mean. Hence the average instantaneous fluid velocity emerging from the pores is decoupled into:

$$\eta\mathbf{v}=\eta(\hat{\mathbf{v}}+\mathring{\mathbf{v}}) \quad (29)$$

where $\hat{\mathbf{v}}$ and $\mathring{\mathbf{v}}$ are regarded as the advection and dispersion velocities respectively.

By virtue of spatial averaging mean values, the integrands in (27) and (28a) yield a product of deviation terms (i.e. part of the integrand embedded between the parentheses) and an average instantaneous term. Thus in view of (29) and the above statement, equation (28a) is rewritten to its mean value $\bar{I}_1$.

$$\bar{I}_1 = \int_{t_1}^{t_2} \int_V \left[\bar{\bar{\eta}} \frac{DC}{Dt} - \mathbf{A} \cdot \nabla(\eta \mathbf{d} \cdot \nabla c)\right] |\mathbf{v}| \, \mathrm{d}V \, \mathrm{d}t \qquad (30)$$

By taking into account the role of deviation that is related to the term within the parentheses, one can sense that $\mathbf{d}$ indicates a measurement of dispersion toward the direction ∇c. Thus in view of (25) and with regard to the directions related to the velocities, $\mathbf{d}$ is changed to a tensor $\boldsymbol{d}^*$ that can take the form of:

$$d_{ij}^* = \alpha_{ijmn} \cdot \frac{v_m v_n}{|\mathbf{v}|} \qquad m, n, i, j = 1, 2, 3 \qquad (31)$$

Hence with regard to (29), (31), equations (30) and (27) are written in their functional form

$$\bar{I}_0 = \int_A \eta c \boldsymbol{d}^* \cdot \mathbf{n} \, \mathrm{d}A \qquad (32)$$

$$\bar{I}_1 = \int_{t_1}^{t_2} \int_V \left[\bar{\bar{\eta}} \frac{DC}{Dt} - \mathbf{A} \cdot \nabla(\eta \boldsymbol{d}^* \nabla c)\right] |\mathbf{v}| \, \mathrm{d}V \, \mathrm{d}t \qquad (33)$$

Thus it is now left to follow the procedure described before (equation (13b) and above it) upon the application of the variation.

Carrying out the variation operator on (33) yields a governing integral form for mass transport, part of its integrand is the so-called dispersion advection differential equation:

$$\eta \frac{DC}{Dt} = \nabla[\eta \boldsymbol{d}^* \nabla c] \qquad (34)$$

CONCLUSION

A stationary principle is described as a working tool to achieve equations of motion and transport when applied to the accumulation of energy, momentum and their appropriate dissipation functions, respectively.

Such a representation allows the intake of information allocated at certain points throughout the time-space domain. Applying spatial-time interpolation functions and proceeding with the variation, a system of algebraic equations that of searching an extremum for the governing functional is obtained. The simultaneous solution of the set for the entire time duration terminates the need for step-by-step time procedure.

The theory combined with a numerical technique (e.g. the F.E. method) can be studied in the future as a procedure for different applications. Also it is suggested that the theory can serve as a minimum criterion to check the validity of a numerical solution.

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