Theory of Elasticity ME 6102
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Biharmonic Functions in Polar Coordinates

ϕ_i	$\ln r$	θ	n = 0	n = -1		n = 1		$\lvert n \rvert \geq 2$	
			1	$r^{-1}\cos\theta$	$r^{-1}\sin\theta$	$r\cos\theta$	$r\sin\theta$	$r^{\pm n}\cos(\pm n\theta)$	$r^{\pm n}\sin(\pm n\theta)$
1	0	*	0	(1)	(1)	(1)	(1)	(n)	(n)
	$\ln r$	0	1	$r^{-1}\cos\theta$	$r^{-1}\sin\theta$	$r\cos\theta$	$r\sin\theta$	$r^{\pm n}\cos(\pm n\theta)$	$r^{\pm n}\sin(\pm n\theta)$
$r\sin\theta$	(1)	*	(1)	(n)	(n)	(n)	(n)	(n)	(n)
	$r\ln r\sin\theta$	$r\theta\sin\theta$	$r\sin\theta$						
$r\cos\theta$	(1)	*	(1)	(n)	(n)	(n)	(n)	(n)	(n)
	$r\ln r\cos\theta$	$r\theta\cos\theta$	$r\cos\theta$						
r^2	0	*	0	(1)	(1)	(1)	(1)	(n)	(n)
	$r^2\ln r$	$r^2\theta$	r^2	$r\cos\theta$	$r\sin\theta$	$r^3\cos\theta$	$r^3\sin\theta$	$r^{\pm n+2}\cos(\pm n\theta)$	$r^{\pm n+2}\sin(\pm n\theta)$

$\phi^{(0)}$ -a function of r only

$$\phi^{(0)}(r) = a_0 + b_0 \ln r + c_0 r^2 + d_0 r^2 \ln r$$

$$\sigma_{rr} = \frac{b_0}{r^2} + 2c_0 + d_0(1 + 2\ln r)$$

$$\sigma_{\theta\theta} = -\frac{b_0}{r^2} + 2c_0 + d_0(3 + 2\ln r)$$

$$\tau_{r\theta} = 0$$

$$\nabla^2\phi = \sigma_{rr} + \sigma_{\theta\theta} = 4c_0 + 4d_0(1 + \ln r)$$

$$U = \frac{1}{E}\left[-(1-\nu)\frac{b_0}{r} + 2(1-\nu)c_0 r + 2(1-\nu)d_0 r \ln r - (1+\nu)d_0 r\right] +$$

$$+ \alpha_2 \cos\theta + \alpha_3 \sin\theta$$

$$V = \frac{1}{E}(4d_0 r\theta) - \alpha_1 r - \alpha_2 \sin\theta + \alpha_3 \cos\theta$$

$$\alpha_1 = \omega^0$$

$$\alpha_2 = a_x^0$$

$$\alpha_3 = a_y^0$$

$\phi^{(*)}$ -a function of explicit θ

$$\phi^{(*)} = a_*\theta + b_*r^2\theta + c_*r\theta \sin\theta + d_*r\theta\cos\theta$$

$$\sigma_{rr} = 2b_*\theta + 2c_*\frac{\cos\theta}{r} - 2d_*\frac{\sin\theta}{r}$$

$$\sigma_{\theta\theta} = 2b_*\theta$$

$$\tau_{r\theta} = \frac{a_*}{r^2} - b_*$$

$$\nabla^2\phi = \sigma_{rr} + \sigma_{\theta\theta} = 4b_*\theta + 2c_*\frac{\cos\theta}{r} - 2d_*\frac{\sin\theta}{r}$$

$$U = \frac{1}{E}[2(1-\nu)b_*r\theta + (1-\nu)c_*\theta\sin\theta + 2c_*\ln r\cos\theta +$$

$$+ (1-\nu)d_*\theta\cos\theta - 2d_*\ln r\sin\theta] + \alpha_2\cos\theta + \alpha_3\sin\theta$$

$$V = \frac{1}{E}[-(1+\nu)\frac{a_*}{r} + (3-\nu)b_*r - 4b_*r\ln r - (1+\nu)c_*\sin\theta$$

$$- 2c_*\ln r\sin\theta + (1-\nu)c_*\theta\cos\theta - (1+\nu)d_*\cos\theta$$

$$- 2d_*\ln r\cos\theta - (1-\nu)d_*\theta\sin\theta] - \alpha_1 r - \alpha_2\sin\theta + \alpha_3\cos\theta$$

$$\alpha_1 = \omega^0$$

$$\alpha_2 = a_x^0$$

$$\alpha_3 = a_y^0$$

$\phi^{(1)}$ - functions of $\sin\theta$ or $\cos\theta$

$$\phi^{(1)} = [a_1 r + \frac{b_1}{r} + c_1 r^3 + d_1 r \ln r] \begin{matrix}\sin\theta \\ \cos\theta\end{matrix}$$

$$\sigma_{rr} = [\frac{-2b_1}{r^3} + 2c_1 r + \frac{d_1}{r}] \begin{matrix}\sin\theta \\ \cos\theta\end{matrix}$$

$$\sigma_{\theta\theta} = [\frac{2b_1}{r^3} + 6c_1 r + \frac{d_1}{r}] \begin{matrix}\sin\theta \\ \cos\theta\end{matrix}$$

$$\tau_{r\theta} = [-\frac{2b_1}{r^3} + 2c_1 r + \frac{d_1}{r}] \begin{matrix}(-\cos\theta) \\ \sin\theta\end{matrix}$$

$$\nabla^2\phi = \sigma_{rr} + \sigma_{\theta\theta} = [8c_1 r + \frac{2d_1}{r}] \begin{matrix}\sin\theta \\ \cos\theta\end{matrix}$$

$$U = \frac{1}{E}\{[(1+\nu)\frac{b_1}{r^2} + (1-3\nu)c_1 r^2 + (1+\nu)d_1 +$$

$$+ (1-\nu)d_1 \ln r] \begin{matrix}\sin\theta \\ \cos\theta\end{matrix} - [2d_1\theta] \begin{matrix}\cos\theta \\ (-\sin\theta)\end{matrix}\} + \alpha_2 \cos\theta + \alpha_3 \sin\theta$$

$$V = \frac{1}{E}(-(1+\nu)\frac{b_1}{r^2} - (5+\nu)c_1 r^2 + (1-\nu)d_1 \ln r] \begin{matrix}\cos\theta \\ (-\sin\theta)\end{matrix} +$$

$$+ [2d_1\theta] \begin{matrix}\sin\theta \\ \cos\theta\end{matrix}) - \alpha_1 r - \alpha_2 \sin\theta + \alpha_3 \cos\theta$$

$$\alpha_1 = \omega^0$$

$$\alpha_2 = a_x^0$$

$$\alpha_3 = a_y^0$$

$\phi^{(n)}$ - functions of $\sin n\theta$ or $\cos n\theta$ for $n>2$

$$\phi^{(n)} = [a_n r^n + b_n r^{-n} + c_n r^{n+2} + d_n r^{-n+2}] \begin{matrix}\sin n\theta \\ \cos n\theta\end{matrix}$$

$$\sigma_{rr} = [a_n(n-n^2)\, r^{n-2} - b_n(n+n^2) r^{-n-2} +$$

$$+ c_n(2+n-n^2) r^n + d_n(2-n-n^2) r^{-n}] \begin{matrix}\sin n\theta \\ \cos n\theta\end{matrix}$$

$$\sigma_{\theta\theta} = [a_n(n^2-n) r^{n-2} + b_n(n+n^2) r^{-n-2} +$$

$$+ c_n(2+3n+n^2) r^n + d_n(2-3n+n^2) r^{-n}] \begin{matrix}\sin n\theta \\ \cos n\theta\end{matrix}$$

$$\tau_{r\theta} = [a_n(n^2-n) r^{n-2} - b_n(n+n^2) r^{-n-2} +$$

$$+ c_n(n+n^2) r^n + d_n(n-n^2) r^{-n}] \begin{matrix}(-\cos n\theta) \\ \sin n\theta\end{matrix}$$

$$\nabla^2\phi = \sigma_{rr} + \sigma_{\theta\theta} = [4c_n(1+n) r^n + 4d_n(1-n) r^{-n}] \begin{matrix}\sin n\theta \\ \cos n\theta\end{matrix}$$

$$U = \frac{1}{E}\{ -a_n(1+\nu) n r^{n-1} + b_n(1+\nu) n r^{-n-1} +$$

$$+ c_n[4-(1+\nu)(2+n)] r^{n+1} + d_n[4-(1+\nu)(2-n)] r^{-n-1} \} \begin{matrix}\sin n\theta \\ \cos n\theta\end{matrix}$$

$$+ \alpha_2 \cos\theta + \alpha_3 \sin\theta$$

$$V = \frac{1}{E}\{-a_n(1+\nu) n r^{n-1} - b_n(1+\nu) n r^{-n-1} - c_n[4+(1+\nu)n] r^{n+1} +$$

$$+ d_n[4-(1+\nu)n] r^{-n+1} \} \begin{matrix}\cos n\theta \\ (-\sin n\theta)\end{matrix} - \alpha_1 r - \alpha_2 \sin\theta + \alpha_3 \cos\theta$$

$$\alpha_1 = \omega^0 \qquad \alpha_2 = a_x^0 \qquad \alpha_3 = a_y^0$$